

Intermittency and multifractality: A case study via parabolic stochastic PDEs*

Davar Khoshnevisan
University of Utah

Kunwoo Kim
University of Utah

Yimin Xiao
Michigan State University

Last update: March 20, 2015

Abstract

Let ξ denote space-time white noise, and consider the following stochastic partial differential equations: (i) $\dot{u} = \frac{1}{2}u'' + u\xi$, started identically at one; and (ii) $\dot{Z} = \frac{1}{2}Z'' + \xi$, started identically at zero. It is well known that the solution to (i) is intermittent, whereas the solution to (ii) is not. And the two equations are known to be in different universality classes.

We prove that the tall peaks of both systems are multifractals in a natural large-scale sense. Some of this work is extended to also establish the multifractal behavior of the peaks of stochastic PDEs on $\mathbb{R}_+ \times \mathbb{R}^d$ with $d \geq 2$. G. Lawler has asked us if intermittency is the same as multifractality. The present work gives a negative answer to this question.

As a byproduct of our methods, we prove also that the peaks of the Brownian motion form a large-scale monofractal, whereas the peaks of the Ornstein–Uhlenbeck process on \mathbb{R} are multifractal.

Throughout, we make extensive use of the macroscopic fractal theory of M.T. Barlow and S.J. Taylor [3, 4]. We expand on aspects of the Barlow–Taylor theory, as well.

Keywords: Intermittency, multifractality, macroscopic/large-scale Hausdorff dimension, stochastic partial differential equations.

AMS 2010 subject classification: Primary. 60H15; Secondary. 35R60, 60K37.

1 Introduction, and main results

The principle aim of this article is to answer the following question that was posed to us by Gregory Lawler in January of 2012 [private communication]:

*Research supported in part by NSF grant DMS-1307470.

Is “intermittency” the same property as “multifractality”? We will argue below that the short answer is no. Among other things, it follows that the macroscopic analysis of disordered systems can unravel a great deal more complexity than its microscopic counterparts (as compared with the theory of Paladin et al [34], for example).

The two quoted terms, “intermittency” and “multifractality,” appear also in the title of this article. They are meant to be understood as informal descriptions of behavior that is commonly observed in a vast array of complex scientific problems in which there are infinitely-many natural length scales. This sort of behavior is common in, but not limited to, problems in full-blown turbulence.

Intermittency is a well-defined mathematical property which we recall first. Multifractality will be treated afterward, and will require more effort. In fact, one of the novel parts of this paper is to set up a mathematical framework within which we can understand macroscopic multifractality in a way that is meaningful in the present context.

In order to motivate some of the results of this paper let us consider the eigenvalue problem for the random heat operator,

$$H := \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial x^2} - M \quad (t > 0, x \in \mathbb{R}), \quad (1.1)$$

acting on space-time functions $f = f(t, x)$ such that $f(0, x) \equiv 1$ for all $x \in \mathbb{R}$, say. Here, M denotes the random multiplication operator, defined via

$$(Mf)(t, x) := f(t, x)\xi(t, x) \quad (t > 0, x \in \mathbb{R}), \quad (1.2)$$

where ξ is a space-time white noise. Because ξ is a random Schwartz-type distribution, and not a nice classically-defined random process, the operator M needs to be understood in integrated form: As a Wiener-integral map when f is nonrandom; and more generally as a Walsh-type stochastic integral operator when f is a predictable random field.

It is well known, and also easy to see, that the spectrum of H is all of \mathbb{R} , and that the eigenfunction of H that corresponds to eigenvalue $\lambda \in \mathbb{R}$ is $\exp(-\lambda t)u(t, x)$, where $u = u(t, x)$ solves the *parabolic Anderson model*, which we may understand rigorously as the solution to the following Walsh-type stochastic partial differential equation [39]:

$$\begin{cases} \dot{u} = \frac{1}{2}u'' + u\xi & \text{on } (0, \infty) \times \mathbb{R}, \\ \text{subject to } u(0, x) = 1 & \text{for all } x \in \mathbb{R}. \end{cases} \quad (\text{PAM})$$

It is well known that the parabolic Anderson model (PAM) has a unique solution which is a predictable random field that is continuous in both variables t and x .¹ Moreover, there exist finite and positive constants U_0, U, L_0, L such that

$$L_0^k \exp(Lk^3t) \leq \mathbb{E}(|u(t, x)|^k) \leq U_0^k \exp(Uk^3t), \quad (1.3)$$

¹The term predictable is used as in Walsh [39], and refers to predictability with respect to the filtration generated by the white noise ξ .

valid uniformly for all real numbers $t > 0$, $k \geq 2$, and $x \in \mathbb{R}$. [In fact, these moments do not depend on the value of x .] See for example Chapter 5 of [28] for a self-contained account. As a consequence, the moment Lyapunov exponents of the solution u are strictly positive and finite, where the k th moment Lyapunov exponent of u is defined as

$$\lambda(k) := \lim_{t \rightarrow \infty} t^{-1} \log \mathbb{E} (|u(t, 0)|^k), \quad (1.4)$$

for every real number $k > 0$.

One can devise a subadditivity argument in order to show that these Lyapunov exponents exist. It is widely believed that the following *Kardar formula* holds [25–27]:²

$$\lambda(k) = k(k^2 - 1)/24 \quad \text{for all real numbers } k > 0. \quad (1.5)$$

The *a priori* bounds (1.3) and convexity considerations together imply that

$$k \mapsto k^{-1} \lambda(k) \quad \text{is strictly increasing on } [2, \infty). \quad (1.6)$$

This property is known as *intermittency* and was referred to earlier on in a broader context.

It is possible to argue that because of intermittency the solution to the parabolic Anderson model (PAM) has very tall peaks on infinitely-many different length scales. We have not yet described very precisely what it means to “have tall peaks on infinitely-many different scales.” Still, the adjective “multifractal” is supposed to reflect the presence of such behavior. The interested reader can find different heuristic accounts of how intermittency might explain multifractality in the Introductions of Bertini and Cancrini [5] and Carmona and Molchanov [7]. Chapter 7 of Khoshnevisan [28, §7.1] yields a related but slightly more precise explanation. A recent paper by Gibbon and Titi [20] contains an account of how, and why, intermittency and other attempts at describing “multifractality” arise naturally in a large number of multi-scale problems in science.

Now that we recalled “intermittency,” within context, we begin to propose a mathematical model of “multifractality” that is general enough for our later needs, but concrete enough that it is amenable to exact analysis. It turns out to be helpful to not begin with a too-concrete setting such as stochastic PDEs. Therefore, let $X := \{X(t)\}_{t \in \mathbb{R}_+^n}$ be a real-valued stochastic process with continuous trajectories. We are interested in saying that X has tall peaks that are “multifractal” in a large-scale sense. In order to do this, let us posit that the tall peaks of the stochastic process X are described by a *gauge function* $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. By this we mean that g is a nonrandom and increasing function such that $\lim_{r \rightarrow \infty} g(r) = \infty$ and

$$\limsup_{\|t\| \rightarrow \infty} \frac{|X(t)|}{g(\|t\|)} = 1 \quad \text{a.s.} \quad (1.7)$$

²Khoshnevisan [28, Chapter 6] has shown that $L \geq k(k^2 - 1)/24$ for every integer $k \geq 2$. Thus, in particular, $\lambda(k) \geq k(k^2 - 1)/24$ for all integers $k \geq 2$. Borodin and Corwin [6] have found the first completely-rigorous proof of (1.5) in the closely-related case that $k \geq 2$ is an integer and the initial value of (PAM) is replaced by δ_0 .

That is, we assume that there exists a *non-random* gauge function g which describes the largest possible scale on which we can compute the tall peaks of X in a macroscopic sense. Let us fix the gauge function g in our mind, and then consider the random set,

$$\mathcal{P}_X(\gamma) := \left\{ t \in \mathbb{R}_+^n : \|t\| > c_0, \frac{|X(t)|}{g(\|t\|)} > \gamma \right\}, \quad (1.8)$$

where $\gamma > 0$ is a tuning parameter and c_0 is a fixed positive constant. We may think of every point $t \in \mathcal{P}_X(\gamma)$ as a *tall peak* of the process X —suitably normalized by g —*viewed in length scale* γ .

Because $\mathcal{P}_X(\gamma_1) \subseteq \mathcal{P}_X(\gamma_2)$ when $\gamma_1 > \gamma_2$, we might expect the existence of a phase transition that does not depend on our prior choice of c_0 . This is correct, and not difficult to explain. In fact, the growth condition (1.7) implies that there exists a uniquely-defined transition point at $\gamma = 1$: $\mathcal{P}_X(\gamma)$ is a.s. unbounded when $\gamma < 1$ whereas $\mathcal{P}_X(\gamma)$ is a.s. bounded when $\gamma > 1$. Motivated by this simple observation, let us introduce the following.

Definition 1.1. Choose and fix a gauge function g and consider the tall peaks of X , as defined in (1.8). We say that the tall peaks of X are *multifractal* when there exist infinitely-many length scales $\gamma_1 > \gamma_2 > \cdots > 0$ such that with probability one,

$$\text{Dim}_H(\mathcal{P}_X(\gamma_{i+1})) < \text{Dim}_H(\mathcal{P}_X(\gamma_i)) \quad \text{for all } i \geq 1. \quad (1.9)$$

If, on the other hand,

$$\text{Dim}_H(\mathcal{P}_X(\gamma)) = \begin{cases} 1 & \text{whenever } \gamma < 1, \\ 0 & \text{whenever } \gamma > 1, \end{cases} \quad (1.10)$$

then we say that the tall peaks of X are *monofractal*.³

In the preceding, $\text{Dim}_H(E)$ denotes the macroscopic Hausdorff dimension of $E \subset \mathbb{R}^n$. This notion is due to Barlow and Taylor [3, 4], and will be reviewed in the next section. For the time being, it suffices to know only that $\text{Dim}_H(E)$ is a real number between 0 and n , and describes the large-scale geometry of $E \subset \mathbb{R}_+^n$ in a way that parallels how the usual notions of fractal dimension try to describe small-scale geometry.

In order to see how our definition of multifractality can be used in stochastic PDEs, let us return to the stochastic heat equation, (PAM). A deep theorem of Mueller [30] asserts that

$$u(t, x) > 0 \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}, \quad (1.11)$$

³In principle, it can happen that the tall peaks of X are neither multifractal nor monofractal. That is, when (1.9) holds for finitely-many $\gamma_1 > \gamma_2 > \cdots > \gamma_n$. It might be interesting to construct non-trivial examples of stochastic processes whose tallest peaks are of this latter type. We aren't aware of any natural examples at this time.

outside a single event of probability zero. Because the natural logarithm is strictly monotonic, the tall peaks of the random function $x \mapsto u(t, x)$ are in one-to-one correspondence with the tall peaks of $x \mapsto X_t(x)$ at all times $t > 0$, where

$$X_t(x) := \left(\frac{32}{9t}\right)^{1/3} \log u(t, x). \quad (1.12)$$

Conus et al [12] have proved that for all $t > 0$,

$$0 < \limsup_{x \rightarrow \infty} \frac{X_t(x)}{g(x)} < \infty \quad \text{a.s., where } g(x) := (\log_+ x)^{2/3}. \quad (1.13)$$

More recently, Chen [8] has found the following improvement to (1.13): For all $t > 0$,

$$\limsup_{x \rightarrow \infty} \frac{X_t(x)}{g(x)} = 1 \quad \text{a.s.} \quad (1.14)$$

In other words, $g(x) := (\log_+ x)^{2/3}$ is a natural gauge function for measuring the tall peaks of the random height function $x \mapsto X_t(x)$. The particular normalization of X in (1.12) is there to merely ensure that the lim sup is one, which matches the form of (1.14) with that of (1.7). Having said this, we are ready to mention one of the results of this work.

Theorem 1.2. *Consider the gauge function $g(x) := (\log_+ x)^{2/3}$ and, for all $t > 0$, define $X_t = \{X_t(x), x \in \mathbb{R}\}$ as in (1.12). Then the tall peaks of X_t are multifractal with probability one for all $t > 0$. In fact, for every $t, \gamma > 0$,*

$$\text{Dim}_{\text{H}}(\mathcal{P}_{X_t}(\gamma)) = 1 - \gamma^{3/2} \quad \text{a.s.}, \quad (1.15)$$

where $\text{Dim}_{\text{H}}(E) < 0$ means that E is bounded.

Theorem 1.2 shows that the tall peaks of the solution u to (PAM) are multifractal in the sense described more precisely in Theorem 1.2. And we can think of (1.15) as a description of large-scale “multifractal spectrum” of the peaks of the random field u . Theorem 1.2 is a consequence of a much deeper result which says, using informal language, that the tall peaks of the solution to every known multifractal nonlinear stochastic PDE [18] are generically multifractal. See Theorem 7.2 and Corollary 7.3 below.

Earlier we mentioned that u is intermittent in the sense of (1.6). Together with Theorem 1.2, this assertion says that intermittency and multifractality can coexist. Next we describe a *non-intermittent* system whose tall peaks are also multifractal. Consequently, non-intermittency and multifractality can also coexist, whence follows the negative answer to Professor Lawler’s question with which we began.

Consider the linear stochastic heat equation,

$$\begin{cases} \dot{Z} = \frac{1}{2}Z'' + \xi & \text{on } (0, \infty) \times \mathbb{R}, \\ \text{subject to } Z(0, x) = 0 & \text{for all } x \in \mathbb{R}. \end{cases} \quad (\text{HE})$$

It is well known that a solution exists, is unique, and is a continuous centered Gaussian random field. It is also possible to prove that the moment Lyapunov exponents of Z are all zero, and so Z is not an intermittent process.⁴

Also, one can prove fairly easily that for all $t > 0$ the following holds with probability one:

$$\limsup_{x \rightarrow \infty} \frac{Z(t, x)}{g(x)} = (t/\pi)^{1/4} \quad \text{a.s., where } g(x) := (2 \log_+ x)^{1/2}. \quad (1.16)$$

A “steady state” version of this fact appears in print, for example, in Collela and Lanford [10, Theorem 1.1(c)]. And the fact itself follows by specializing an even earlier, very general, theory of Pickands [36]; see also Qualls and Watanabe [37].

Let

$$X_t(x) := (\pi/t)^{1/4} Z(t, x). \quad (1.17)$$

The following shows that the tall peaks of X_t —hence also those of $Z(t, \cdot)$ —are multifractal even though Z is non-intermittent.

Theorem 1.3. *Consider the gauge function $g(x) := (\log_+ x)^{1/2}$ and, for all $t > 0$, define $X_t = \{X_t(x), x \in \mathbb{R}\}$ as in (1.17). Then the tall peaks of X_t are multifractal with probability one for all $t > 0$. In fact, for every $t, \gamma > 0$,*

$$\text{Dim}_H(\mathcal{P}_{X_t}(\gamma)) = 1 - \gamma^2 \quad \text{a.s.}, \quad (1.18)$$

where $\text{Dim}_H(E) < 0$ means that E is bounded.

Theorems 1.2 and 1.3 are both consequences of two more general theorems about multifractal random fields [Theorems 4.1 and 4.7 below]. Those general theorems have other interesting consequences as well. Let us mention one such result.

Theorem 1.4. *Let B denote a one-dimensional Brownian motion and $U(t) := \exp(-t/2)B(e^t)$ an Ornstein–Uhlenbeck process on \mathbb{R} . Then, the tall peaks of B are a monofractal, whereas those of U are multifractal in the following sense: For every $\gamma > 0$,*

$$\text{Dim}_H \left\{ s \geq e^e : \frac{B(s)}{(2s \log \log s)^{1/2}} \geq \gamma \right\} = \begin{cases} 1 & \text{if } \gamma \leq 1, \\ 0 & \text{if } \gamma > 1, \end{cases} \quad (1.19)$$

almost surely, whereas

$$\text{Dim}_H \left\{ s \geq e^e : \frac{U(s)}{(2 \log s)^{1/2}} \geq \gamma \right\} = 1 - \gamma^2, \quad (1.20)$$

almost surely, where we recall $\text{Dim}_H(E) < 0$ means that E is bounded.

⁴This can be deduced from inspecting the proof of Theorem 2.1 of ref. [18], for example.

Theorem 1.4 will be proved in two parts: (1.19) is proved below in Theorem 3.2; and (1.20) is proved in Theorem 5.1.

We have included a final §8 wherein we state and prove a suitable variation of Theorem 1.2, in which the stochastic partial differential equation (PAM) is replaced by a similar-looking object with u'' replaced by Δu , the Laplacian, the latter acting on a space variable $x \in \mathbb{R}^d$ in place of \mathbb{R} . And space-time white noise is replaced by a centered Gaussian noise that is white in time and suitably correlated in space to ensure the existence and uniqueness of a well-tempered solution. Among other things, such equations are well-known models that play a role in the large-scale structure of the universe; see §8 for more details.

Let us conclude the Introduction with a word on notation. From now on we will write $\phi_t(x)$ and $\phi(t, x)$ interchangeably, depending on which is more convenient, for any space-time function [or even generalized function/noise] ϕ . This choice ought not inconvenience the reader, since ϕ_t is *never* used to denote the time derivative of ϕ ; rather, it is the standard probabilistic notation for describing the time evolution of ϕ .

2 Dimension and density

Let us begin by recalling the Barlow–Taylor definition of the macroscopic dimension [3, 4] $\text{Dim}_H E$ of a set $E \subseteq \mathbb{R}^d$.

Define, for all integers $n \geq 0$,

$$\mathcal{V}_n := [-e^n, e^n]^d, \quad \mathcal{S}_0 := \mathcal{V}_0, \quad \text{and} \quad \mathcal{S}_{n+1} := \mathcal{V}_{n+1} \setminus \mathcal{V}_n. \quad (2.1)$$

We might sometimes refer to \mathcal{S}_n as the *n*th shell in \mathbb{R}^d .

An important idea of Barlow–Taylor [3, 4]—see also Naudts [32] for a precursor to this idea—is to construct a family of Hausdorff-type contents on each shell, and then use the totality of those contents in order to define a family of Hausdorff-type contents on all of \mathbb{R}^d . Once this is done, a notion of macroscopic Hausdorff dimension presents itself quite naturally.

Definition 2.1. Let \mathcal{B} denote the collection of all sets of the form

$$Q(x, r) := [x_1, x_1 + r) \times \cdots \times [x_d, x_d + r), \quad (2.2)$$

as $x := (x_1, \dots, x_d)$ ranges in \mathbb{R}^d and r in $(0, \infty)$. If $Q := Q(x, r)$ is an element of \mathcal{B} , then we may refer to Q as an *upright box* with *southwest corner* x and *sidelength* $\text{side}(Q) := r$.

Choose and fix some number $c_0 > 0$, and define for every set $E \subseteq \mathbb{R}^d$, all real numbers $\rho > 0$, and each integer $n \geq 0$,

$$\nu_\rho^n(E) := \inf \sum_{i=1}^m \left(\frac{\text{side}(Q_i)}{e^n} \right)^\rho, \quad (2.3)$$

where the infimum is taken over all upright boxes Q_1, \dots, Q_m of side $\geq c_0$ that cover $E \cap \mathcal{S}_n$. We may think of $\nu_\rho^n(E)$ as the restriction to the *n*th shell \mathcal{S}_n of the scaled ρ -dimensional *Hausdorff content* of E .

Definition 2.2. The Barlow–Taylor *macroscopic Hausdorff dimension* of $E \subseteq \mathbb{R}^d$ is

$$\text{Dim}_H E := \text{Dim}_H(E) := \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} \nu_{\rho}^n(E) < \infty \right\}. \quad (2.4)$$

By (2.4), any bounded set $E \subseteq \mathbb{R}^d$ has $\text{Dim}_H E = 0$. We will leave the following simple fact as exercise for the interested reader.

Lemma 2.3. *The numerical value of $\text{Dim}_H E$ does not depend on $c_0 > 0$.*

Thus, we could choose $c_0 = 2$ or $c_0 = e/\sqrt{2}$, in our definition of ν_{ρ}^n , without affecting the value of $\text{Dim}_H E$. It is important to point out though that $c_0 = 0$ can lead to a different value of $\text{Dim}_H E$.

In order to see why we are ruling out the possibility of $c_0 = 0$, let us define for all $E \subseteq \mathbb{R}^d$ and $r > 0$ a new set $E^z(r) \subseteq \mathbb{Z}^d$ as follows:

$$E^z(r) := \{x \in \mathbb{Z}^d : E \cap Q(x, r) \neq \emptyset\}. \quad (2.5)$$

The notation is basically due to Barlow and Taylor, who observed the following [3, Lemma 6.1] but stated it using slightly different language: For all $E \subseteq \mathbb{R}^d$ and $r > 0$,

$$\text{Dim}_H E = \text{Dim}_H(E^z(r)), \quad (2.6)$$

where Dim_H on the right-hand side of (2.6) is the discrete Hausdorff dimension on \mathbb{Z}^d of Barlow and Taylor [3, 4]. In other words, because we chose only covers of E that have side $\geq c_0 > 0$, the local structure of E does not affect the value of its macroscopic Hausdorff dimension. Put yet in another way, this shows that the Barlow–Taylor definition of $\text{Dim}_H E$ *quantifies the large-scale geometry of E* , without obstruction by the microscopic structure of the set E .

Since the particular value of $c_0 > 0$ does not matter, from now on we follow the choice of Barlow and Taylor [3, 4], and set

$$c_0 = 1. \quad (2.7)$$

The following is a macroscopic counterpart of a familiar result about microscopic Hausdorff dimension.

Lemma 2.4. *Suppose $f : E \rightarrow \mathbb{R}^p$ is a Lipschitz function on $E \subseteq \mathbb{R}^d$ and satisfies the growth condition $\liminf_{x \in E, |x| \rightarrow \infty} |f(x)|/|x| > 0$. Then,*

$$\text{Dim}_H f(E) \leq \text{Dim}_H E. \quad (2.8)$$

In particular, if $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is bi-Lipschitz on $E \subseteq \mathbb{R}^d$; that is, there exists a positive constant $L \geq 1$ such that

$$L^{-1}|x - y| \leq |f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in E.$$

Then

$$\text{Dim}_H f(E) = \text{Dim}_H E. \quad (2.9)$$

Proof. The proof is similar to that of the same assertion for ordinary Hausdorff dimension: For every $\rho > \dim_{\mathbb{H}} E$, we can find upright boxes $Q_{j,n} \in \mathcal{B}$ —indexed by $1 \leq j \leq m_n$, $n \geq 1$ —with $\text{side}(Q_{j,n}) \geq 1$ such that:

- (i) $E \cap \mathcal{S}_n \subseteq \cup_{j=1}^{m_n} Q_{j,n}$ for all $n \geq 1$; and
- (ii) $\sum_{n=1}^{\infty} \sum_{j=1}^{m_n} (\text{side}(Q_{j,n})/e^n)^\rho < \infty$.

It is clear that $f(E) \subset \cup_{n=1}^{\infty} \cup_{j=1}^{m_n} f(Q_{j,n})$. Since f is Lipschitz on E we can find a finite and positive constant q such that

$$|f(x) - f(y)| \leq q|x - y| \quad \text{for all } x, y \in E. \quad (2.10)$$

Without loss of generality, we may assume that $q \geq 1$ is an integer; otherwise, we replace q by $1 + [q]$ everywhere. In particular, it follows that every $f(Q_{j,n})$ can be covered with an upright box whose sidelength is between 1 and $q \text{side}(Q_{j,n})$. In addition, the growth condition on f implies that there is a constant $\varepsilon > 0$ such that $f(x) \geq \varepsilon e^n$ for every $x \in E \cap \mathcal{S}_n$. This implies that

$$\sum_{n=0}^{\infty} \nu_{\rho}^n(f(E)) \leq \sum_{n=0}^{\infty} q^{\rho} \sum_{j=1}^{m_n} \left(\frac{\text{side}(Q_{j,n})}{\varepsilon e^n} \right)^{\rho} \quad \text{for all } n \geq 1, \quad (2.11)$$

whence $\sum_{n=0}^{\infty} \nu_{\rho}^n(f(E)) < \infty$ by (ii). This proves that $\dim_{\mathbb{H}} f(E) \leq \rho$ for all $\rho > \dim_{\mathbb{H}} E$ and implies (2.8).

Finally, the bi-Lipschitz condition implies that both f and its inverse f^{-1} on $f(E)$ satisfy the growth condition. Hence, we make two appeals to the first part of Lemma 2.4, once for f and once for f^{-1} , to see that (2.9) holds. \square

Lemma 2.4 is the large-scale/macrosopic analogue of the following well-known fact: If $f : E \rightarrow \mathbb{R}$ is locally Lipschitz continuous, then

$$\dim_{\mathbb{H}} f(E) \leq \dim_{\mathbb{H}} E, \quad (2.12)$$

where “ $\dim_{\mathbb{H}}$ ” denotes the usual [microscopic] Hausdorff dimension in \mathbb{R}^d . Let us, however, observe that (2.8) does not hold when f is only Lipschitz. For example, set $f(x) := \ln(x)$ for $x \geq 1$ and $E = \exp(\mathbb{N})$ to see that

$$\dim_{\mathbb{H}} f(E) = \dim_{\mathbb{H}} \mathbb{N} = 1 > 0 = \dim_{\mathbb{H}}(\exp(\mathbb{N})). \quad (2.13)$$

In the above, $\mathbb{N} := \{1, 2, \dots\}$ denotes as usual the set of all natural numbers. We will present an interesting example of E in Remark 5.2 below which shows that (2.8) does hold for $f(x) := \ln(x)$ even though f does not satisfies the growth condition in Lemma 2.4.

Next, let us mention a technical estimate, which is a “density theorem.” The following is a large-scale analogue of the classical Frostman lemma, and basically rephrases Theorem 4.1(a) of Barlow and Taylor [3] in a slightly different form that is more convenient for us.

Lemma 2.5 (A Frostman-type lemma). *Choose and fix an integer $n \geq 1$, and suppose $E \subset \mathcal{S}_n$ is a Borel set in \mathbb{R}^d . Let μ denote a finite non-zero Borel measure on E , and define for all $\rho \geq 0$,*

$$K_\rho := \sup \left\{ \frac{\mu(Q)}{[\text{side}(Q)]^\rho} : Q \in \mathcal{B}, Q \subset \mathcal{S}_n, \text{side}(Q) \geq 1 \right\}. \quad (2.14)$$

Then, $\nu_\rho^n(E) \geq K_\rho^{-1} e^{-n\rho} \mu(E)$.

Remark 2.6. The constant K_ρ of Lemma 2.5 typically depends on n as well, and is always finite and positive.

Let ν be a Borel measure on \mathbb{R}^d . For all $E \subseteq \mathbb{R}^d$ let $\text{Den}_\nu E$ denotes the *upper density* of E with respect to ν . That is,

$$\text{Den}_\nu E := \text{Den}_\nu(E) := \limsup_{t \rightarrow \infty} \frac{\nu(E \cap [-t, t]^d)}{(2t)^d}. \quad (2.15)$$

The following describes an easy-to-verify sufficient condition for a set E in \mathbb{R}^d to have full macroscopic Hausdorff dimension.

Lemma 2.7. *Let ν denote either the Lebesgue measure on \mathbb{R}^d or counting measure on a sublattice $\varepsilon\mathbb{Z}^d$ of \mathbb{R}^d for some $\varepsilon > 0$. Then, for all Borel sets $E \subseteq \mathbb{R}^d$, if $\text{Den}_\nu E > 0$, then $\text{Dim}_H E = d$.*

Proof. We will prove the lemma only in the case that ν denotes the Lebesgue measure. When ν is the counting measure on $\varepsilon\mathbb{Z}^d$, the result is proved in almost exactly the same way.

Barlow and Taylor [3, 4] have introduced another large-scale notion of Hausdorff dimension, which we write as follows:

$$\underline{\text{Dim}}_H E := \underline{\text{Dim}}_H(E) := \inf \left\{ \rho > 0 : \lim_{n \rightarrow \infty} \nu_\rho^n(E) = 0 \right\}. \quad (2.16)$$

It is easy to verify that $\underline{\text{Dim}}_H E \leq \text{Dim}_H E$ for all $E \subseteq \mathbb{R}^d$; therefore we might think of $\underline{\text{Dim}}_H$ as the *lower Hausdorff dimension*, in the macroscopic sense. Our goal is to prove the following somewhat stronger statement:

$$\text{Den}_\nu E > 0 \quad \Rightarrow \quad \underline{\text{Dim}}_H E \geq d. \quad (2.17)$$

It follows from (2.17) that $\text{Dim}_H E \geq d$. Since $\text{Dim}_H E \leq \text{Dim}_H(\mathbb{R}^d) = d$ (see Barlow and Taylor [4, Example 4.1]), (2.17) completes the proof.

For every real number $a > 1$ and integers $n \geq 0$ define

$$\begin{aligned} \mathcal{V}_n(a) &:= [-a^n, a^n]^d, \\ \mathcal{S}_0(a) &:= \mathcal{V}_0(a), \\ \mathcal{S}_{n+1}(a) &:= \mathcal{V}_{n+1}(a) \setminus \mathcal{V}_n(a). \end{aligned} \quad (2.18)$$

Note, in particular, that $\mathcal{S}_n = \mathcal{S}_n(e)$ and $\mathcal{V}_n = \mathcal{V}_n(e)$. Define for every set $E \subset \mathbb{R}^d$, all real numbers $\rho > 0$, and each integer $n \geq 0$,

$$\nu_\rho^n(E; a) := \min \sum_{i=1}^m \left(\frac{\text{side}(Q_i)}{a^n} \right)^\rho, \quad (2.19)$$

where the minimum is taken over all upright boxes Q_1, \dots, Q_m of side ≥ 1 that cover $E \cap \mathcal{S}_n(a)$.

Barlow and Taylor [4, p. 127] have remarked that their macroscopic and lower Hausdorff dimensions do not depend on the choice of a ; in particular,

$$\underline{\text{Dim}}_{\text{H}} E = \inf \left\{ \rho > 0 : \lim_{n \rightarrow \infty} \nu_\rho^n(E; a) = 0 \right\} \quad \text{for all } a > 1. \quad (2.20)$$

In fact, the original construction of Barlow and Taylor [3, 4] is similar to ours, but with $a = 2$ and not $a = e$, as is the case here.

We may define a Borel measure μ on E by setting

$$\mu(G) := |E \cap G| \quad \text{for all } G, \quad (2.21)$$

where $|\dots|$ denotes the Lebesgue measure. Clearly,

$$K_d := \sup \left\{ \frac{\mu(Q)}{r^d} : Q \in \mathcal{B}, Q \subseteq \mathcal{S}_n, \text{side}(Q) \in [1, r], r \geq 1 \right\} \leq 1. \quad (2.22)$$

Furthermore,

$$\mu(\mathcal{V}_n(a)) \leq |\mathcal{V}_n(a)| = 2^d a^{nd} \quad \text{for all } n \geq 0, \quad (2.23)$$

and

$$\mu(\mathcal{V}_{n+1}(a)) \geq 2^d (\text{Den}_\nu E - \delta_n) a^{(n+1)d}, \quad (2.24)$$

for infinitely-many integers $n \geq 1$, where $\{\delta_n\}_{n=1}^\infty$ is a sequence that satisfies $\lim_{n \rightarrow \infty} \delta_n = 0$. Therefore,

$$\mu(\mathcal{S}_{n+1}(a)) = \mu(\mathcal{V}_{n+1}(a)) - \mu(\mathcal{V}_n(a)) \geq (a^d \text{Den}_\nu E - \delta_n - 1) 2^d a^{nd}, \quad (2.25)$$

for infinitely-many integers $n \geq 0$. It is easy to adapt Lemma 2.5 to an analogous statement about $\nu_\rho^n(E; a)$ for all choices of a [not only $a = e$]. That endeavor shows us that $\nu_d^n(E; a) \geq a^d \text{Den}_\nu E - \delta_n - 1$ for infinitely-many integers $n \geq 0$. Therefore,

$$\limsup_{n \rightarrow \infty} \nu_d^n(E; a) > 0, \quad (2.26)$$

provided that we choose $a > \max\{1, (\text{Den}_\nu E)^{-1/d}\}$. This and (2.20) together imply (2.17) and hence the lemma. \square

3 Peaks of Brownian motion

Consider the set of times, after time $t = \exp(e)$ [say], at which the Brownian motion has LIL-type behavior. That is, let us fix some parameter $\gamma > 0$, and consider the random set

$$\mathcal{P}_B(\gamma) := \left\{ s \geq e^e : \frac{B(s)}{(2s \log \log s)^{1/2}} \geq \gamma \right\}. \quad (3.1)$$

We are using a notation that is consistent with that in (1.7) and (1.8), where $g(x) := (2x \log_+ \log_+ x)^{1/2}$ is the gauge function that comes to us naturally from the standard law of the iterated logarithm for Brownian motion. That is, the following:

Proposition 3.1 (The law of the iterated logarithm). *With probability one:*

1. $\mathcal{P}_B(\gamma)$ is unbounded a.s. when $\gamma \leq 1$;
2. $\mathcal{P}_B(\gamma)$ is bounded a.s. when $\gamma > 1$.

Actually, the standard textbook form of the LIL refers only to the case that $\gamma < 1$ and $\gamma > 1$. The critical case $\gamma = 1$ follows from Motoo's work [29, Example 2]. The following shows that when $\mathcal{P}_B(\gamma)$ is unbounded, that is when $\gamma \leq 1$, it is a macroscopic fractal of dimension one. [Of course, $\mathcal{P}_B(\gamma)$ does not have a remarkable macroscopic structure when $\gamma > 1$.]

Theorem 3.2. *Assertion (1.19) of Theorem 1.4 holds; that is,*

$$\text{Dim}_H \mathcal{P}_B(\gamma) = 1 \quad \text{a.s. for all } \gamma \leq 1. \quad (3.2)$$

In the subcritical case where $\gamma < 1$, Theorem 3.2 follows immediately from Lemma 2.7 and the next statement.

Proposition 3.3 (Strassen [38]). *Let ν denote the Lebesgue measure on \mathbb{R}^d . Then the following assertions are valid a.s.:*

1. $\text{Den}_\nu \mathcal{P}_B(\gamma) > 0$ when $\gamma < 1$; and
2. $\text{Den}_\nu \mathcal{P}_B(1) = 0$.

Proof. One can easily adapt a result of Strassen [38, eq. (11)] about random walks to a statement about linear Brownian motion in order to see that with probability one,

$$\text{Den}_\nu \mathcal{P}_B(\gamma) = 1 - \exp \left\{ -4 \left(\frac{1}{\gamma^2} - 1 \right) \right\}, \quad (3.3)$$

as long as $\gamma \leq 1$. This does the job. \square

Proposition 3.3 shows that Theorem 3.2 is interesting mostly in the critical case. In the critical case, the random set $\mathcal{P}_B(1)$ is comprised of tall peaks of maximum possible asymptotic height. And Theorem 3.2 shows that the set of tall peaks of critical height has full dimension although it has zero density [Proposition 3.3].

With the preceding remarks in mind, let us consider the following random Borel measure that is supported in $\mathcal{P}_B(1)$:

$$\mu(G) := |\mathcal{P}_B(1) \cap G| \quad \text{for all Borel sets } G \subseteq [4, \infty), \quad (3.4)$$

where $|\cdots|$ denotes the 1-dimensional Lebesgue measure. The following is the key step in the proof of Theorem 3.2.

Proposition 3.4. $\sum_{n=4}^{\infty} e^{-n} \mu(\mathcal{S}_n) = \infty$ a.s.

We will begin our proof of Proposition 3.4 shortly. But first, let us apply this proposition in order to establish Theorem 3.2.

Proof of Theorem 3.2. The preceding remarks tell us that we need to only consider the critical case, $\gamma = 1$. Because $\mu[x, x+r) \leq r$, it follows that $K_1 = 1$, where K_ρ was defined in (2.14). Lemma 2.5 and Proposition 3.4 together imply that

$$\sum_{n=4}^{\infty} \nu_1^n(\mathcal{P}_B(1) \cap \mathcal{S}_n) \geq \sum_{n=4}^{\infty} e^{-n} \mu(\mathcal{S}_n) = \infty \quad \text{a.s.} \quad (3.5)$$

In particular, it follows that $\text{Dim}_H \mathcal{P}_B(1) \geq 1$ a.s., which is the desired result. \square

In order to derive Proposition 3.4, let us consider the events

$$\mathcal{E}_t := \left\{ \omega \in \Omega : B(t)(\omega) > (2t \log \log t)^{1/2} \right\} \quad \text{for all } t \geq 4. \quad (3.6)$$

It is easy to see from l'Hôpital's rule that if X has the standard normal distribution, then $\mathbb{P}\{X > z\}$ is to within a multiplicative constant of $z^{-1} \exp(-z^2/2)$ uniformly for all $z \geq 1$. The following is a consequence of this fact and the strict positivity and the continuity of the Gaussian density function: There exists a finite constant $c > 1$ such that

$$\frac{1}{c \log t (\log \log t)^{1/2}} \leq \mathbb{P}(\mathcal{E}_t) \leq \frac{c}{\log t (\log \log t)^{1/2}} \quad \text{for all } t \geq 4. \quad (3.7)$$

Next we estimate $\mathbb{P}(\mathcal{E}_t | \mathcal{E}_s)$ for various choices of $s < t$. The first quantifies the well-known qualitative fact that \mathcal{E}_s and \mathcal{E}_t are approximately independent when $t \gg s$.

Lemma 3.5. *For all real numbers $t > s > 4$,*

$$t > 4s(\log \log s)(\log \log t) \quad \Rightarrow \quad \mathbb{P}(\mathcal{E}_t | \mathcal{E}_s) \leq c \mathbb{P}(\mathcal{E}_t), \quad (3.8)$$

where $c \in (0, \infty)$ does not depend on (s, t) .

Proof. We recall the following well-known bound, which is essentially Lemma 1.5 of Orey and Pruitt [33]: If U and V are jointly distributed as a bivariate normal with common mean zero, common variance one, and covariance ρ , then there exists a finite constant c such that

$$\mathbb{P}(U > a \mid V > b) \leq c \mathbb{P}\{U > a\} \quad \text{whenever } |\rho| < (ab)^{-1}. \quad (3.9)$$

Next we apply the preceding by setting $U := t^{-1/2}B(t)$, $V := s^{-1/2}B(s)$, $a := (2 \log \log t)^{1/2}$, and $b := (2 \log \log s)^{1/2}$. Note that $\rho = (s/t)^{1/2}$ satisfies $0 < \rho < (ab)^{-1}$ because $t > 4s(\log \log s)(\log \log t)$. The lemma follows from (3.9). \square

When t and s are not too far apart, we do not expect $\mathbb{P}(\mathcal{E}_t \mid \mathcal{E}_s)$ to have the same order of magnitude as $\mathbb{P}(\mathcal{E}_t)$. The following provides us with a quantitative estimate of $\mathbb{P}(\mathcal{E}_t \mid \mathcal{E}_s)$ in this case.

Lemma 3.6. *There exists a finite constant c such that for every $t > s > 4$,*

$$\mathbb{P}(\mathcal{E}_t \mid \mathcal{E}_s) \leq \frac{c}{(\log s)^{(t-s)/(4t)}}. \quad (3.10)$$

Proof. According to Lemma 1.6 of Orey and Pruitt [33], if U and V are jointly distributed as a bivariate normal with common mean zero, common variance one, and covariance ρ , then there exists a finite constant c such that

$$\mathbb{P}(U > a \mid V > a) \leq c \exp\left(-\frac{1}{8}(1 - \rho^2)a^2\right) \quad \text{for all } a \geq 0. \quad (3.11)$$

We apply this inequality with $U := t^{-1/2}B(t)$, $V := s^{-1/2}B(s)$, and $a := (2 \log \log s)^{1/2}$ to find that

$$\mathbb{P}(\mathcal{E}_t \mid \mathcal{E}_s) \leq \mathbb{P}\left(B(t) > (2t \log \log s)^{1/2} \mid \mathcal{E}_s\right) = \mathbb{P}(U > a \mid V > a). \quad (3.12)$$

Thus, the lemma follows from (3.11). \square

We are prepared to verify Proposition 3.4.

Proof of Proposition 3.4. For all $N \geq 4$ define

$$S_N := \sum_{n=4}^N e^{-n} \mu(\mathcal{S}_n). \quad (3.13)$$

We intend to prove that $S_\infty := \lim_{N \rightarrow \infty} S_N$ is infinite a.s. Because $\mathbb{E}S_N = \sum_{n=4}^N e^{-n} \int_{e^{n-1}}^{e^n} \mathbb{P}(\mathcal{E}_s) ds$, we may apply (3.7) in order to see that

$$\mathbb{E}S_N \geq c_0 (\log N)^{1/2} \quad \text{for all } N \geq 4, \quad (3.14)$$

where c_0 is a positive constant that does not depend on N . Next, we estimate the variance of S_N . Clearly,

$$\mathbb{E}(S_N^2) \leq Q_1 + Q_2, \quad (3.15)$$

where

$$\begin{aligned} Q_1 &:= 2 \sum_{n=4}^N e^{-2n} \iint_{e^{n-1} < s < t < e^n} \mathbb{P}(\mathcal{E}_s \cap \mathcal{E}_t) ds dt, \\ Q_2 &:= 2 \sum_{4 \leq n < m \leq N} e^{-n-m} \int_{e^{n-1}}^{e^n} ds \int_{e^{m-1}}^{e^m} dt \mathbb{P}(\mathcal{E}_s \cap \mathcal{E}_t). \end{aligned} \quad (3.16)$$

The elementary bound $\mathbb{P}(\mathcal{E}_s \cap \mathcal{E}_t) \leq \mathbb{P}(\mathcal{E}_s)$ yields

$$Q_1 \leq 2\mathbb{E}S_N. \quad (3.17)$$

We estimate Q_2 by splitting the double sum according to how much the summation variable m is greater than the summation variable n . Before we hash out the details, let us first note that, according to Lemma 3.6, whenever $m > n \geq 4$,

$$\begin{aligned} & \int_{e^{n-1}}^{e^n} ds \int_{e^{m-1}}^{e^m} dt \mathbb{P}(\mathcal{E}_s \cap \mathcal{E}_t) \\ & \leq c_1 \int_{e^{n-1}}^{e^n} ds \int_{e^{m-1}}^{e^m} dt \mathbb{P}(\mathcal{E}_s) \exp\left(-\frac{t-s}{4t} \log \log s\right) \\ & \leq c_1 \int_{e^{n-1}}^{e^n} ds \int_{e^{m-1}}^{e^m} dt \mathbb{P}(\mathcal{E}_s) \exp\left(-c_2 \frac{t - e^{m-1}}{e^m} \log n\right) \\ & \leq c_1 e^m \int_{e^{n-1}}^{e^n} ds \int_0^\infty dt \mathbb{P}(\mathcal{E}_s) e^{-c_2 t \log n} = \frac{c_3 e^m}{\log n} \int_{e^{n-1}}^{e^n} \mathbb{P}(\mathcal{E}_s) ds, \end{aligned} \quad (3.18)$$

where c_1, c_2, c_3 are finite and positive constants that do not depend on (n, m) . Consequently,

$$\begin{aligned} & \sum_{4 \leq n < m \leq (n + \alpha \log n) \wedge N} e^{-n-m} \int_{e^{n-1}}^{e^n} ds \int_{e^{m-1}}^{e^m} dt \mathbb{P}(\mathcal{E}_s \cap \mathcal{E}_t) \\ & \leq c_3 \alpha \sum_{n=4}^N e^{-n} \int_{e^{n-1}}^{e^n} \mathbb{P}(\mathcal{E}_s) ds = c_3 \alpha \mathbb{E}S_N, \end{aligned} \quad (3.19)$$

for all $\alpha > 0$. We emphasize that c_1, c_2 , and c_3 are finite constants that do not depend on (N, α) , and α is, so far, an arbitrary parameter.

Next we observe that there exists $\alpha > 0$ large enough such that for all integers $m, n \geq 4$,

$$m > n + \alpha \log n \quad \Rightarrow \quad e^{m-1} > 4e^n (\log n)(\log m). \quad (3.20)$$

We will choose α to be this particular value, both in the preceding and in what follows. In this case, it is then easy to see that $t > 4s(\log \log s)(\log \log t)$ for

every $s \in \mathcal{S}_n$ and $t \in \mathcal{S}_m$, as long as $n, m \geq 4$ are arbitrary integers that satisfy $m > n + \alpha \log n$. Thus, Lemma 3.5 ensures that, for this particular choice of α ,

$$\begin{aligned} \sum_{4 \leq n < n + \alpha \log n} \sum_{n < m \leq N} e^{-n-m} \int_{e^{n-1}}^{e^n} ds \int_{e^{m-1}}^{e^m} dt \mathbb{P}(\mathcal{E}_s \cap \mathcal{E}_t) \\ \leq c_4 \left(\sum_{n=4}^N e^{-n} \int_{e^{n-1}}^{e^n} \mathbb{P}(\mathcal{E}_s) ds \right)^2 = c_4 |\mathbb{E} S_N|^2, \end{aligned} \quad (3.21)$$

where c_4 is a finite constant that does not depend on n . Thanks to (3.16), (3.19), and (3.21), it follows that $Q_2 \leq c_3 \alpha \mathbb{E} S_N + c_4 |\mathbb{E} S_N|^2$, uniformly in all $N \geq 4$. Thus, it follows from (3.14) and (3.17) that

$$\mathbb{E}(S_N^2) = O(|\mathbb{E} S_N|^2) \quad \text{as } N \rightarrow \infty. \quad (3.22)$$

Since $\mathbb{E} S_N \uparrow \infty$ as $N \rightarrow \infty$ [see (3.14)] and $S_\infty \geq S_N$ for all N ,

$$\mathbb{P}\{S_\infty = \infty\} \geq \liminf_{N \rightarrow \infty} \mathbb{P}\{S_N \geq \tfrac{1}{2} \mathbb{E} S_N\} \geq \frac{1}{4} \liminf_{N \rightarrow \infty} \frac{|\mathbb{E} S_N|^2}{\mathbb{E}(S_N^2)} > 0, \quad (3.23)$$

thanks to the Paley–Zygmund inequality [35, Lemma γ], which states that if Z is in $L^2(\mathbb{P})$ and $\|Z\|_\infty > 0$, then

$$\mathbb{P}\{Z > \tfrac{1}{2} \mathbb{E} Z\} \geq \frac{1}{4} \frac{(\mathbb{E} Z)^2}{\mathbb{E}(Z^2)}. \quad (3.24)$$

An appeal to the Hewitt–Savage 0–1 law completes the proof. \square

4 General bounds

Let $X := \{X_t\}_{t \in T}$ be a real-valued random field with continuous trajectories, where $T \subseteq \mathbb{R}^d$ is either one of the 2^d standard closed orthants of \mathbb{R}^d , or T is \mathbb{R}^d itself.

For all real numbers $b \in (0, \infty)$ we can define

$$\mathbf{c}(b) := - \limsup_{z \rightarrow \infty} z^{-b} \sup_{t \in T} \log \mathbb{P}\{X_t > z\}, \quad (4.1)$$

and

$$\mathbf{C}(b) := - \liminf_{z \rightarrow \infty} z^{-b} \inf_{t \in T} \log \mathbb{P}\{X_t > z\}. \quad (4.2)$$

Of course, $0 \leq \mathbf{c}(b) \leq \mathbf{C}(b) \leq \infty$ for all $b > 0$.

Define $|t|$ to be the ℓ^∞ -norm of $t \in \mathbb{R}^d$; that is,

$$|t| := \max_{1 \leq i \leq d} |t_i| \quad \text{for all } t := (t_1, \dots, t_d) \in \mathbb{R}^d. \quad (4.3)$$

Then it should be intuitively clear that, under mild conditions on large-scale smoothness and asymptotic pairwise independence of X , one ought to be able to prove that

$$0 < \limsup_{|t| \rightarrow \infty} \frac{X_t}{(\log |t|)^{1/b}} < \infty \quad \text{a.s.}, \quad (4.4)$$

provided additionally that $0 < \mathbf{c}(b) \leq \mathbf{C}(b) < \infty$ for a certain special value of $b \in (0, \infty)$. In other words, we might expect that if $0 < \mathbf{c}(b) \leq \mathbf{C}(b) < \infty$, then the tall peaks of the process X are typically gauged, to within a constant, by the function $t \mapsto (\log |t|)^{1/b}$.

The main results of this section are two general macroscopic Hausdorff dimension estimates; see Theorems 4.1 and 4.7 below. The first theorem describes conditions, similar to those outlined earlier, which ensure the upper bound in (4.4), and also bound from above the macroscopic Hausdorff dimension of the set of times that X_t exceeds a [correct] constant multiple of $(\log |t|)^{1/b}$. The second theorem turns out to be a much more subtle result that produces matching lower bounds for the dimension of these exceedance times. Thus, we begin with a general upper bound for the Hausdorff dimension of the tall peaks of a stochastic process.

Theorem 4.1 (A general upper bound). *Suppose that there exists $b \in (0, \infty)$ such that $\mathbf{c}(b) > 0$ and for all $\gamma \in (0, d)$,*

$$\sup_{w \in T} \mathbb{P} \left\{ \sup_{t \in [w, w+1)} X_t > \left(\frac{\gamma}{\mathbf{c}(b)} \log s \right)^{1/b} \right\} \leq s^{-\gamma+o(1)} \quad \text{as } s \rightarrow \infty. \quad (4.5)$$

Then,

$$\limsup_{|t| \rightarrow \infty} \frac{X_t}{(\log |t|)^{1/b}} \leq \left(\frac{d}{\mathbf{c}(b)} \right)^{1/b} \quad \text{a.s.} \quad (4.6)$$

Furthermore,

$$\text{Dim}_{\mathbf{H}} \left\{ t \in T : |t| > \exp(e), X_t \geq \left(\frac{\gamma}{\mathbf{c}(b)} \log |t| \right)^{1/b} \right\} \leq d - \gamma, \quad (4.7)$$

for all $\gamma \in (0, d)$.

Proof. It suffices to prove the result in the case that $T = \mathbb{R}_+^d$. The other cases, including $T = \mathbb{R}^d$, follow from this after making small adjustments.

The stated limsup result is a more-or-less standard exercise in the Borel–Cantelli lemma, and the upper bound on the dimension follows from a routine covering argument, the likes of which are familiar for bounding the microscopic Hausdorff of a random set. We include the proof for the sake of completeness.

Let us write $\text{Dim}_{\mathbf{H}} \Sigma < 0$ when Σ is a bounded set. We plan to prove that (4.7) holds for all $\gamma > 0$; (4.6) follows immediately from this formulation of (4.7). From now on we choose and fix an arbitrary $\gamma > 0$.

Our goal is to prove that

$$\text{Dim}_H \Lambda_u \leq d - \gamma \quad \text{a.s.}, \quad (4.8)$$

where

$$\Lambda_u := \left\{ t \in \mathbb{R}_+^d : |t| > \exp(e), X_t \geq \left(\frac{\gamma}{c(b)} \log |t| \right)^{1/b} \right\}. \quad (4.9)$$

Condition (4.5) ensures that for all $\varepsilon \in (0, \gamma)$ there exists a finite constant K_ε such that for all $m := (m_1, \dots, m_d) \in \mathbb{R}_+^d$ that satisfy $\|m\| > \exp(e)$,

$$\begin{aligned} \mathbb{P}\{\Lambda_u \cap Q(m, 1) \neq \emptyset\} &\leq \mathbb{P}\left\{ \sup_{t \in Q(m, 1)} X_t \geq \left(\frac{\gamma}{c(b)} \log |m| \right)^{1/b} \right\} \\ &\leq \frac{K_\varepsilon}{|m|^{\gamma-\varepsilon}}. \end{aligned} \quad (4.10)$$

Therefore, we can cover $\Lambda_u \cap \mathcal{S}_n$ by upright boxes of sidelength $r \equiv 1$ in order to see that for all $\rho > 0$ and $n \geq 0$ large,

$$\begin{aligned} \mathbb{E} [\nu_\rho^n(\Lambda_u)] &\leq \sum_{\substack{m \in \mathbb{Z}_+^d : \\ Q(m, 1) \subset \mathcal{S}_n}} \left(\frac{1}{e^n} \right)^\rho \mathbb{P}\{\Lambda_u \cap Q(m, 1) \neq \emptyset\} \\ &\leq K_\varepsilon e^{-n\rho} \sum_{\substack{m \in \mathbb{Z}_+^d : \\ Q(m, 1) \subset \mathcal{S}_n}} |m|^{-\gamma+\varepsilon}. \end{aligned} \quad (4.11)$$

Whenever $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+^d$ are such that $Q(m, 1)$ lies entirely in \mathcal{S}_n , then it must be that $|m| \geq \exp(n-1)$. Since \mathcal{S}_n contains at most $\text{const} \cdot \exp(nd)$ upright boxes of sidelength one, it follows that

$$\mathbb{E} [\nu_\rho^n(\Lambda_u)] \leq \text{const} \cdot e^{-n[\rho+\gamma-\varepsilon-d]}, \quad (4.12)$$

for all $n \geq 1$ sufficiently large. In particular,

$$\mathbb{E} \left[\sum_{n=0}^{\infty} \nu_\rho^n(\Lambda_u) \right] < \infty \quad \text{if } \rho > d - \gamma + \varepsilon. \quad (4.13)$$

This proves that $\text{Dim}_H \Lambda_u \leq \rho$ a.s. for all $\rho > d - \gamma + \varepsilon$. Send $\rho \downarrow d - \gamma + \varepsilon$ and then $\varepsilon \downarrow 0$, in this order, to deduce (4.8) and hence the theorem. \square

We now move on to a perhaps more interesting study of lower bounds for Dim_H of high peaks of X ; more specifically, our next result shows that one can sometimes obtain good lower bounds on the macroscopic dimension in the statement of Theorem 4.1.

Consider, for every $\gamma \in (0, d)$, the random set

$$\Lambda_\ell := \left\{ t \in T : |t| > \exp(e), X_t \geq \left(\frac{\gamma}{C(b)} \log t \right)^{1/b} \right\}. \quad (4.14)$$

We plan to show that, under some conditions on the process X ,

$$\text{Dim}_H \Lambda_\ell \geq d - \gamma \quad \text{a.s.}, \quad (4.15)$$

thus obtaining a complimentary bound to that of Theorem 4.1.

The standard way to obtain lower bounds on the ordinary Hausdorff dimension of a set is to find a smooth measure on that set; see Lemma 2.5 for example. The said smooth measure is typically “uniquely canonical,” and readily guessed when the set in question has good local structure. The remaining work is in determining the exact order of the smoothness of the canonical measure.

In the present setting of macroscopic dimension, one can in fact prove that the random set Λ_ℓ does *not* support a “uniquely canonical” measure. Instead we have to use a different sort of argument.

Before we proceed we need to develop a few general facts about macroscopic Hausdorff dimension. We will also introduce some notation that will be used throughout the remainder of this section.

Definition 4.2. Let Π_n be a finite collection of points in \mathbb{R}^d for every integer $n \geq 0$. Given a real number $\theta \in (0, 1)$, we say that $\{\Pi_n\}_{n=0}^\infty$ is a θ -skeleton of \mathbb{R}^d if there exists an integer $N = N(\theta)$ such that:

1. For every $n \geq N$,

$$\bigcup_{x \in \Pi_n} Q(x, e^{\theta n}) \subset \mathcal{S}_n; \quad (4.16)$$

2. If x and y are distinct points in Π_n for some $n \geq N$, then

$$Q(x, e^{\theta n}) \cap Q(y, e^{\theta n}) = \emptyset; \quad \text{and} \quad (4.17)$$

3. There exists a constant $a = a(d, \theta) \in (0, 1)$ such that

$$a e^{nd(1-\theta)} \leq |\Pi_n| \leq a^{-1} e^{nd(1-\theta)}, \quad (4.18)$$

where “ $|\dots|$ ” denotes cardinality.

Given some $\theta \in (0, 1)$, \mathbb{R}^d has uncountably-many θ -skeletons. From now on, we choose and fix one such choice, and denote it by $\Pi(\theta) := \{\Pi_n(\theta)\}_{n=0}^\infty$. For instance, we might wish to opt for the following construction, to be explicit:

$$\Pi_n(\theta) := A_n(\theta) \times \dots \times A_n(\theta) \quad [d \text{ times}]; \quad (4.19)$$

where

$$A_n(\theta) := \bigcup_{\substack{0 \leq j \leq e^{n(1-\theta)+1} - e^n \\ j \in \mathbb{Z}}} \{e^n + j e^{\theta n}\} \quad (4.20)$$

Other constructions are also possible. The property of $\Pi(\theta)$ that we are after this: $\Pi_n(\theta)$ is basically a “near-optimal $e^{\theta n}$ -packing” of \mathcal{S}_n for all large n . Part 2 of Definition 4.2 codifies the precise notion of “packing” and Part 3 makes precise our “near-optimal” sense.

Definition 4.3. Let $E \subseteq \mathbb{R}^d$ be a set and choose and fix some real number $\theta \in (0, 1)$. We say that E is θ -thick if there exists an integer $M = M(\theta)$ such that

$$E \cap Q(x, e^{\theta n}) \neq \emptyset, \quad (4.21)$$

for all $x \in \Pi_n(\theta)$ and $n \geq M$.

In words, θ -thick sets are “approximately self-similar sets.”

We are ready to present one of the novel technical statements of this section.

Proposition 4.4. *If $E \subset \mathbb{R}^d$ is θ -thick for some $\theta \in (0, 1)$, then*

$$\text{Dim}_H E \geq d(1 - \theta). \quad (4.22)$$

Proposition 4.4 presents us with a strategy for obtaining a lower bound for $\text{Dim}_H F$ that can be different from a Frostman-type method [Lemma 2.5]. The following is an immediate consequence of Proposition 4.4.

Corollary 4.5. *If $F \subset \mathbb{R}^d$ has a θ -thick subset E for some $\theta \in (0, 1)$, then $\text{Dim}_H F \geq d(1 - \theta)$.*

In principle, our definition of θ -thickness depends on our *a priori* choice of a θ -skeleton $\Pi(\theta)$. We are not aware of any example where the choice matters very much. But even if the choice did matter, we can interpret Corollary 4.5 as saying that we can always obtain a lower bound on $\text{Dim}_H F$ by finding a θ -skeleton $\Pi(\theta)$ of \mathbb{R}^d and a set $E \subset F$ that is θ -thick with respect to our choice of skeleton. In that case, $\text{Dim}_H F \geq d(1 - \theta)$.

It remains to prove Proposition 4.4.

Proof of Proposition 4.4. Using the notation of Definitions 4.2 and 4.3 we can find a finite number of points $x_{1,n}, \dots, x_{L_n,n} \in \mathcal{S}_n$ such that

$$|x_{i,n} - x_{j,n}| \geq e^{\theta n} \quad \text{for all } 1 \leq i \neq j \leq L_n, \quad (4.23)$$

where in fact $L_n := L_n(\theta) := |\Pi_n(\theta)|$. In particular, (4.18) ensures that

$$ae^{nd(1-\theta)} \leq L_n \leq a^{-1}e^{nd(1-\theta)} \quad \text{for all } n. \quad (4.24)$$

It might help to recall that $a \in (0, 1)$ is independent of the parameters (i, j, n) of this discussion.

Even though F might not support a particularly-natural measure, the following defines a very natural locally finite measure μ on E :

$$\mu(F) := \sum_{n=M}^{\infty} \sum_{j=1}^{L_n} \mathbf{1}_F(x_{j,n}), \quad (4.25)$$

for all $F \subseteq \mathbb{R}^d$.

Because $\mu(\mathcal{S}_n) = L_n$, the total μ -mass of the n th shell satisfies

$$ae^{nd(1-\theta)} \leq \mu(\mathcal{S}_n) \leq a^{-1}e^{nd(1-\theta)} \quad \text{for all } n \geq M. \quad (4.26)$$

Since μ is a measure on E , we plan to use the measure μ in Lemma 2.5 in order to find a lower bound for $\text{Dim}_H(E)$. With this aim in mind, we next establish an upper bound for $\mu(Q)$ for every uprightbox Q whose sidelength is at least one, with the sole provision that $Q \subset \mathcal{S}_n$ for some $n \geq M$ that is sufficiently large.

Let us consider an arbitrary upright box $Q(z, r)$ of sidelength $r \geq 1$ such that $Q(z, r)$ lies entirely in \mathcal{S}_n for some large enough integer $n \geq M$. Elementary properties of the Euclidean space \mathbb{R}^d imply that there exists a positive integer $k \leq 2^d$ together with k points z_1, \dots, z_k from the collection $\{x_{1,n}, \dots, x_{L_n,n}\}$ such that

$$Q(z, r) \subseteq \bigcup_{j=1}^k Q(z_j, r). \quad (4.27)$$

Therefore,

$$\mu(Q(z, r)) \leq \sum_{j=1}^k \mu(Q(z_j, r)) = k \leq 2^d. \quad (4.28)$$

This shows that for all $\rho > 0$,

$$K_\rho := \sup \left\{ \frac{\mu(Q)}{r^\rho} : Q \in \mathcal{B}, Q \subset \mathcal{S}_n, \text{side}(Q) \in [1, r], r \geq 1 \right\} \leq 2^d. \quad (4.29)$$

On the other hand, if $r \geq e^{\theta n}$, then an upright box $Q(z, r)$ can contain at most $(1 + re^{-\theta n})^d$ -many points in $\Pi_n(\theta)$. Now,

$$(1 + re^{-\theta n})^d \leq 2^d r^d e^{-d\theta n}, \quad (4.30)$$

because $r \geq e^{\theta n}$, and $\text{side}(Q(z, r)) \leq e^n$ because $Q(z, r) \subset \mathcal{S}_n$. Therefore, it follows from the construction of the measure μ that

$$\begin{aligned} \sup \left\{ \frac{\mu(Q)}{[\text{side}(Q)]^\rho} : Q \in \mathcal{B}, Q \subset \mathcal{S}_n, \text{side}(Q) \geq e^{\theta n} \right\} \\ \leq 2^d e^{-d\theta n} \sup_{e^n \geq r \geq e^{\theta n}} r^{d-\rho} = 2^d e^{-n[d\theta - d + \rho]} \end{aligned} \quad (4.31)$$

as long as $0 < \rho < d$. Now we compare (4.29) and (4.31), and set $\rho := d(1 - \theta)$ in order to see that $K_{d(1-\theta)} \leq 2^d$, where K_ρ was defined in (4.29); see also (2.14). This bound and (4.26) together yield the following: For all $n \geq M$ sufficiently large,

$$\nu_{d(1-\theta)}^n(E) \geq 2^{-d} e^{-nd(1-\theta)} \mu(\mathcal{S}_n) \geq a 2^{-d}, \quad (4.32)$$

thanks to Lemma 2.5. It follows that $\limsup_{n \rightarrow \infty} \nu_{d(1-\theta)}^n(E) \geq a 2^{-d} > 0$, and hence

$$\text{Dim}_H(E) \geq \underline{\text{Dim}}_H E \geq d(1 - \theta), \quad (4.33)$$

where $\underline{\text{Dim}}_H$ denotes the lower Hausdorff dimension of Barlow and Taylor [3, 4], as was recalled in (2.16). \square

Let us introduce a last piece of notation before we state and prove the general lower bound of this section [Theorem 4.7 below]. That lower bound will be a counterpart to Theorem 4.1.

Definition 4.6. Let \mathcal{I} denote the collection of all independent finite sequences of independent random variables.

Then we have the following general lower bound statement.

Theorem 4.7. *Suppose there exists $b \in (0, \infty)$ such that $C(b) < \infty$. Suppose in addition that there exist $\delta \in (0, 1)$ and an increasing nonrandom measurable function $S : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$n^{-1} \max_{\{t_i\}_{i=1}^m \in \Pi_n(\delta)} \max_{1 \leq j \leq m} \inf_{\{Y_i\}_{i=1}^m \in \mathcal{I}} \log \mathbb{P}\{|S(X_{t_j}) - S(Y_j)| > 1\} \rightarrow -\infty, \quad (4.34)$$

as $n \rightarrow \infty$. Then,

$$\limsup_{t \rightarrow \infty} \frac{X_t}{(\log t)^{1/b}} \geq \left(\frac{d}{C(b)} \right)^{1/b}, \quad (4.35)$$

a.s. Moreover, if $\gamma \in (0, d)$ then

$$\text{Dim}_H \left\{ t \in T : |t| > \exp(e), \frac{X_t}{(\log t)^{1/b}} \geq \left(\frac{\gamma}{C(b)} \right)^{1/b} \right\} \geq d - \gamma \text{ a.s.} \quad (4.36)$$

Remark 4.8. Condition (4.34) is a coupling assumption, and states that if $t_1, \dots, t_m \in T$ have large norms [say, are in \mathcal{S}_n for a large n] and sufficiently far apart [say at least $\exp(\delta n)$ apart], then X_{t_1}, \dots, X_{t_m} are close—say within one unit—to an independent sequence with very high probability. At first glance this might seem to be a technical and complicated condition. We will see in the next few sections that (4.34) is in fact frequently easy to use, particularly in the context of stochastic PDEs. Condition (4.34) can be recast in terms of the “correlation length” of the process X ; see Conus et al [13] for details.

If $\text{Dim}_H G > 0$ then in particular G is unbounded. In this way we see that (4.36) implies that

$$\limsup_{t \rightarrow \infty} \frac{X_t}{(\log t)^{1/b}} \geq \left(\frac{\gamma}{C(b)} \right)^{1/b}, \quad (4.37)$$

a.s. for all $\gamma \in (0, d)$. Let $\gamma \uparrow d$ to deduce (4.35) from (4.36). Thus, we need to derive only (4.36).

Proof of Theorem 4.7. Since $X_t \geq \alpha$ if and only if $S(X_t) \geq S(\alpha)$, we can replace the random field $\{X_t\}_{t \in T}$ by the random field $\{S(X_t)\}_{t \in T}$ throughout the entire statement of the theorem in order to see that for the remainder of the proof we can—and will—assume without incurring any loss in generality that

$$S(x) := x \quad \text{for all } x \in \mathbb{R}. \quad (4.38)$$

In other words, the function S merely plays the role of a change of “scale.”

We plan to prove that the random set Λ_ℓ —defined earlier in (4.14)—a.s. is θ -thick for every $\theta \in (\gamma/d, 1)$ and $\gamma \in (0, d)$. Owing to Proposition 4.4, this proves that

$$\text{Dim}_H \Lambda_\ell \geq d(1 - \theta) \quad \text{a.s. for all } \theta \in \left(\frac{\gamma}{d}, 1\right) \text{ and } \gamma \in (0, d), \quad (4.39)$$

and (4.36) follows. In light of the paragraph that followed the statement of Theorem 4.7, this endeavor completes the proof of Theorem 4.7. Henceforth, we choose and fix two arbitrary numbers $\gamma \in (0, d)$ and $\theta \in (\gamma/d, 1)$. We also hold fixed an arbitrary [small]

$$0 < \delta < \theta - \frac{\gamma}{d}. \quad (4.40)$$

Now we carry out a multi-scale argument. Recall the definition of $\Pi_n(\theta)$. Because of that definition, for all sufficiently large integers $n \gg 1$, we can find points $x_{1,n}, \dots, x_{L_n,n}$ in \mathcal{S}_n such that

$$Q(x_{i,n}, e^{\theta n}) \cap Q(x_{j,n}, e^{\theta n}) = \emptyset \quad \text{when } 1 \leq i \neq j \leq L_n, \quad (4.41)$$

and

$$ae^{nd(1-\theta)} \leq L_n \leq a^{-1}e^{nd(1-\theta)}, \quad (4.42)$$

where $a \in (0, 1)$ depends neither on n nor on the pair (i, j) . For future purposes, we would like to emphasize that as part of the construction of these points we have also the following:

$$\lim_{n \rightarrow \infty} \frac{\log L_n}{n} = d(1 - \theta). \quad (4.43)$$

For all $1 \leq i \leq L_n$ we can find points $z_{1,n}(i), \dots, z_{\ell_n(i),n}(i)$ in $Q(x_{i,n}, e^{\theta n})$ such that

$$|z_{k,n}(i) - z_{l,n}(i)| > e^{\delta n}, \quad (4.44)$$

whenever $1 \leq k \neq l \leq \ell_n(i)$, and

$$be^{nd(\theta-\delta)} \leq \ell_n(i) \leq b^{-1}e^{nd(\theta-\delta)}, \quad (4.45)$$

where $b \in (0, 1)$ depends neither on n nor on the triple (i, k, l) . In fact, a and b depend only on θ and δ , which are held fixed throughout this discussion.

According to (4.34) for all $K > d$ and $1 \leq i \leq L_n$ we can find an independent sequence Y_1, \dots, Y_m of random variables such that

$$\max_{1 \leq j \leq \ell_n(i)} \mathbb{P}\{|X_{z_{j,n}(i)} - Y_j| > 1\} \leq Ke^{-Kn} \quad \text{for all } n \geq K. \quad (4.46)$$

The particular construction of Y_1, \dots, Y_m might—or might not—depend on K and i ; it does not matter. The upshot is the following: Since

$$\begin{aligned} \mathbb{P}\left\{\sup_{t \in Q(x_{i,n}, e^{\theta n})} \frac{X_t}{(\log t)^{1/b}} \leq \left(\frac{\gamma}{C(b)}\right)^{1/b}\right\} \\ \leq \mathbb{P}\left\{\max_{1 \leq j \leq \ell_n(i)} X_{z_{j,n}(i)} \leq \left(\frac{(n+1)\gamma}{C(b)}\right)^{1/b}\right\}, \end{aligned} \quad (4.47)$$

two back-to-back applications of (4.46) show us that

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{t \in Q(x_{i,n}, e^{\theta n})} \frac{X_t}{(\log t)^{1/b}} \leq \left(\frac{\gamma}{C(b)} \right)^{1/b} \right\} \\ & \leq K e^{-Kn} + \prod_{j=1}^{\ell_n(i)} \mathbb{P} \left\{ Y_j \leq 1 + \left(\frac{(n+1)\gamma}{C(b)} \right)^{1/b} \right\} \end{aligned} \quad (4.48)$$

$$\leq K e^{-Kn} + \prod_{j=1}^{\ell_n(i)} \left(\mathbb{P} \left\{ X_{z_{j,n}(i)} \leq 2 + \left(\frac{(n+1)\gamma}{C(b)} \right)^{1/b} \right\} + K e^{-Kn} \right), \quad (4.49)$$

uniformly for all $0 \leq i \leq L_n$ and $n \geq K$. Given an arbitrary $\varepsilon \in (0, 1)$, we can find $K_0 > K$ such that

$$\begin{aligned} \mathbb{P} \left\{ X_{z_j(i)} \leq 2 + \left(\frac{(n+1)\gamma}{C(b)} \right)^{1/b} \right\} & \leq \mathbb{P} \left\{ X_{z_j(i)} \leq \left(\frac{n(\gamma + \varepsilon)}{C(b)} \right)^{1/b} \right\} \\ & \leq 1 - e^{-(\gamma + \varepsilon)n}, \end{aligned} \quad (4.50)$$

uniformly for all $0 \leq i \leq L_n$, $0 \leq j \leq \ell_n(i)$, and $n \geq K_0$. This follows because $C(b) < \infty$.

Because of (4.40), we can—and will—in fact choose ε small enough so that

$$0 < \varepsilon < \frac{d(\theta - \delta) - \gamma}{2}. \quad (4.51)$$

Therefore, there exists $K_1 > K_0$ such that

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{t \in Q(x_{i,n}, e^{\theta n})} \frac{X_t}{(\log t)^{1/b}} \leq \left(\frac{\gamma}{C(b)} \right)^{1/b} \right\} \\ & \leq K e^{-Kn} + \left(1 - e^{-(\gamma + \varepsilon)n} + K e^{-Kn} \right)^{\ell_n(i)} \\ & \leq K e^{-Kn} + \exp \left\{ -\ell_n(i) e^{-(\gamma + 2\varepsilon)n} \right\}, \end{aligned} \quad (4.52)$$

uniformly for all $0 \leq i \leq L_n$ and $n \geq K_1$. [The preceding hinges on the fact that $K > d > \gamma + \varepsilon$.] We may deduce from (4.45) that

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{t \in Q(x_{i,n}, e^{\theta n})} \frac{X_t}{(\log t)^{1/b}} \leq \left(\frac{\gamma}{C(b)} \right)^{1/b} \right\} \\ & \leq K e^{-Kn} + \exp \left\{ -b e^{(d\theta - d\delta - \gamma - 2\varepsilon)n} \right\}, \end{aligned} \quad (4.53)$$

uniformly for all $0 \leq i \leq L_n$ and $n \geq K_1$. Thanks to (4.51) and the fact that $K > d > n^{-1} \log L_n = d(1 - \theta) + o(1)$ —see (4.43)—it follows that

$$\sum_{n=1}^{\infty} \sum_{i=0}^{L_n} \mathbb{P} \left\{ \sup_{t \in Q(x_{i,n}, e^{\theta n})} \frac{X_t}{(\log t)^{1/b}} \leq \left(\frac{\gamma}{C(b)} \right)^{1/b} \right\} < \infty. \quad (4.54)$$

Therefore, the Borel–Cantelli lemma ensures that the following holds for all ω in the probability space that lie outside of a single set of \mathbb{P} -measure zero: For all but a finite number of integers $n \geq 1$,

$$\sup_{t \in Q(x_{i,n}, e^{\theta n})} \frac{X_t(\omega)}{(\log t)^{1/b}} > \left(\frac{\gamma}{C(b)} \right)^{1/b} \quad \text{for all } 0 \leq i \leq L_n. \quad (4.55)$$

Recall the random set Λ_ℓ that was defined earlier in (4.14). We can state the preceding display in another way; namely, that for all but a finite number of integers $n \geq 1$,

$$\Lambda_\ell \cap Q(x_{i,n}, e^{\theta n}) \neq \emptyset \quad \text{for all } 0 \leq i \leq L_n. \quad (4.56)$$

This proves that Λ_ℓ is θ -thick a.s. Proposition 4.4 then shows that

$$\text{Dim}_H \Lambda_\ell \geq d(1 - \theta) \quad \text{a.s.} \quad (4.57)$$

Since $\theta \in (\gamma/d, 1)$ were arbitrary, we let $\theta \downarrow \gamma/d$ to complete the proof. \square

Let us close this section by recalling a well-known general asymptotic evaluation of the tail of the distribution of the supremum of a stationary Gaussian process with a nice covariance function. The result is originally due to Pickands [36, Lemma 2.5], with extra conditions that were removed subsequently by Qualls and Watanabe [37, Theorem 2.1]. Albin and Choi [2] contain a novel elementary proof, together with an indepth discussion of the literature of the subject.

Lemma 4.9 (Pickands). *Let $\{\eta(t)\}_{t \geq 0}$ denote a continuous stationary Gaussian process with $\mathbb{E} \eta(t) = 0$ and $\text{Var} \eta(t) = 1$ for all $t \geq 0$. Suppose that there exist constants $\kappa \in (0, \infty)$ and $\alpha \in (0, 2]$ such that*

$$\text{Cov}[\eta(t), \eta(0)] = 1 - \kappa t^\alpha + o(t^\alpha) \quad \text{as } t \rightarrow 0^+. \quad (4.58)$$

Then,

$$\mathbb{P} \left\{ \sup_{s \in [0, 1]} \eta(s) > x \right\} = \frac{H_\alpha \kappa^{1/\alpha} + o(1)}{(2\pi)^{1/2}} x^{(2-\alpha)/\alpha} e^{-x^2/2} \quad \text{as } x \rightarrow \infty, \quad (4.59)$$

where $H_\alpha \in (0, \infty)$ is a numerical constant that depends only on α .

Remark 4.10. The cited literature also contains the assumption that there exists $h \in (0, \infty)$ such that $\inf_{t \in [0, h]} \text{Cov}[\eta(t), \eta(0)] > 0$. We have omitted it as it is subsumed by the assumed behavior of $\text{Cov}[\eta(t), \eta(0)]$ near $t = 0$.

Remark 4.11. The Pickands constant H_α is itself rather interesting. We follow Pickands [36] and let $\Phi := \{\Phi(t)\}_{t \geq 0}$ denote a centered Gaussian process with

$$\text{Cov}[\Phi(s), \Phi(t)] = s^\alpha + t^\alpha - |t - s|^\alpha. \quad (4.60)$$

The process Φ is a fractional Brownian motion with parameter $\alpha/2$ when $\alpha \in (0, 2)$, and $\Phi(t) = t\zeta$ for a standard normal random variable ζ when $\alpha = 2$. Then, H_α is the following nontrivial limit [which is known to exist as well]:

$$H_\alpha = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\sup_{t \in [0, T]} e^{\Phi(t) - t^\alpha} \right]. \quad (4.61)$$

It is known that $H_1 = 1$ and $H_2 = \pi^{-1/2}$. Other values of H_α are not known. See Harper [22] and the references therein for recent estimates.

5 Peaks of the Ornstein-Uhlenbeck process

For a first, and perhaps simplest, example of the general theory of §4 let us continue to write B for a standard Brownian motion on \mathbb{R} , and define

$$U(t) := e^{-t/2} B(e^t) \quad \text{for all } t \geq 0. \quad (5.1)$$

Then $U := \{U(t)\}_{t \geq 0}$ is an Ornstein–Uhlenbeck process; that is, U is a centered Gaussian diffusion with $\mathbb{E}[U(t)U(s)] = \exp(-|t - s|/2)$ for all $s, t \geq 0$. Thanks to the law of the iterated logarithm for Brownian motion,

$$\limsup_{t \rightarrow \infty} \frac{U(t)}{(2 \log t)^{1/2}} = 1 \quad \text{a.s.} \quad (5.2)$$

Here we consider the exceedance times of U , defined as follows:

$$\mathcal{P}_U(\gamma) := \left\{ t \geq e : \frac{U(t)}{(2 \log t)^{1/2}} \geq \gamma \right\} \quad (\gamma > 0). \quad (5.3)$$

This notation is consistent with the notation in (1.7) and (1.8) with $g(x) := (2 \log_+ x)^{1/2}$.

Because $\mathcal{P}_U(\gamma) = \log \mathcal{P}_B(\gamma)$ —where $\mathcal{P}_B(\gamma)$ was defined in (3.1)—and the natural logarithm is strictly monotone, we see that the random sets $\mathcal{P}_U(\gamma)$ and $\mathcal{P}_B(\gamma)$ are bounded and unbounded together. In particular, Proposition 3.4 implies that $\mathcal{P}_U(\gamma)$ is unbounded [a.s.] if and only if $\gamma \leq 1$; this fact follows also from the integral test of Motoo [29]. Therefore, the following theorem computes the macroscopic Hausdorff dimension of $\mathcal{P}_U(\gamma)$ in all cases of interest.

Theorem 5.1. *Part (1.20) of Theorem 1.4 holds. Namely,*

$$\dim_{\text{H}} \mathcal{P}_U(\gamma) = 1 - \gamma^2 \quad \text{a.s. for all } \gamma \in (0, 1]. \quad (5.4)$$

Remark 5.2. We can compare Theorems 3.2 and 5.1 in order to see that $\dim_{\text{H}} \log \mathcal{P}_B(1) = 0$ a.s. whereas $\dim_{\text{H}} \mathcal{P}_B(1) = 1$ a.s. Equivalently,

$$\dim_{\text{H}} \exp(\mathcal{P}_U(1)) = 1 \neq 0 = \dim_{\text{H}} \mathcal{P}_U(1) \quad \text{a.s.} \quad (5.5)$$

The elegant theory of Weber [40, Theorem 6.1] implies the following closely-related result: With probability one,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |\{0 \leq j \leq e^n : \mathcal{P}_U(\gamma) \cap [j, j+1) \neq \emptyset\}| = 1 - \gamma^2, \quad (5.6)$$

where $|\dots|$ denotes cardinality here. In other words,

$$\text{Dim}_{\text{M}} \mathcal{P}_U(\gamma) = 1 - \gamma^2 \quad \text{a.s.}, \quad (5.7)$$

where Dim_{M} denotes *macroscopic Minkowski dimension*. In the notation of Barlow and Taylor [3, 4], $\text{Dim}_{\text{M}} E$ is the common value of $\dim_{\text{UM}} E$ and $\dim_{\text{LM}} E$, when the two are equal. Since $\text{Dim}_{\text{H}} E \leq \text{Dim}_{\text{M}} E$ for all $E \subseteq \mathbb{R}^d$ [3, Lemma 3.1(i)], Weber's theorem implies half of Theorem 5.1; i.e., $\text{Dim}_{\text{H}} \mathcal{P}_U(\gamma) \leq 1 - \gamma^2$ a.s. As part of proving the converse inequality, we plan to use Theorems 4.1 and 4.7 in order to obtain both inequalities at the same time.

Proof of Theorem 5.1. We apply Theorems 4.1 and 4.7 with $b = 2$, $X_t := U(t)$ for all $t \geq 0$, and $S(x) := x$ for all $x \in \mathbb{R}$. Since $\{U(t)\}_{t \geq 0}$ is stationary, we can see from an elementary bound on the tails of the Gaussian law that $c(2) = C(2) = 2$.

Because

$$\text{Cov}(U(t), U(0)) = e^{-t/2} = 1 - \frac{t}{2} + o(t) \quad \text{as } t \rightarrow 0, \quad (5.8)$$

Pickands's lemma [Lemma 4.9] implies the maximal inequality (4.5), and our Theorem 4.1 then shows that $\text{Dim}_{\text{H}} \mathcal{P}_U(\gamma) \leq 1 - \gamma^2$ a.s. for all $\gamma \in (0, 1)$.

In order to prove the reverse inequality, let us note that if $t_1 < \dots < t_m$ are in \mathcal{S}_n and $t_{i+1} - t_i \geq \exp\{\delta n\}$ for all $1 \leq i \leq m$, then we may set

$$Y_i := e^{-t_i/2} [B(e^{t_i}) - B(e^{t_{i-1}})] \quad (1 \leq i \leq m), \quad (5.9)$$

with $t_0 := e^n$. It is easy to see that the Y_i 's are independent and

$$\max_{1 \leq i \leq m} \mathbb{E} \left(|U(t_i) - Y_i|^2 \right) = e^{-(t_i - t_{i-1})} \leq \exp \{-e^{\delta n}\}. \quad (5.10)$$

Consequently, a standard bound on the tails of Gaussian laws implies that

$$\max_{1 \leq i \leq m} \mathbb{P} \{|U(t_i) - Y_i| > 1\} \leq 2 \exp \left(-\frac{1}{2} \exp \{e^{\delta n}\} \right). \quad (5.11)$$

Of course, this proves that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \max_{1 \leq i \leq m} \log \mathbb{P}\{|U(t_i) - Y_i| > 1\} = -\infty, \quad (5.12)$$

with room to spare. Hence, Condition (4.34) is verified since the Y_i 's are independent. We can apply Theorem 4.7—with γ replaced by γ^2 —in order to deduce that $\text{Dim}_{\text{H}} \mathcal{P}_U(\gamma) \geq 1 - \gamma^2$ a.s. for all $\gamma \in (0, 1)$. This completes the proof. \square

6 Peaks of the linear heat equation

Now we move on to examples that are perhaps more interesting. Consider the linear stochastic heat equation,

$$\dot{Z}_t(x) = \frac{1}{2}Z_t''(x) + \xi_t(x) \quad (x \in \mathbb{R}, t > 0), \quad (6.1)$$

subject to $Z_0(x) := 0$ [say], where ξ denotes space-time white noise. That, ξ is a totally-scattered centered Gaussian noise with

$$\text{Cov}(\xi_t(x), \xi_s(y)) = \delta_0(s - t)\delta_0(x - y) \quad \text{for } s, t \geq 0 \text{ and } x, y \in \mathbb{R}. \quad (6.2)$$

It is well known—see Walsh [39, Chapter 3] for example—that there exists a unique integral solution to the stochastic PDE (6.1), and that solution has the following representation in terms of a Wiener integral:

$$Z_t(x) = \int_{(0,t) \times \mathbb{R}} p_{t-s}(y - x) \xi(ds dy) \quad (t > 0, x \in \mathbb{R}), \quad (6.3)$$

where the function $(s, t, x, y) \mapsto p_{t-s}(x - y)\mathbf{1}_{(0,\infty)}(t - s)$ denotes the fundamental solution to the heat operator,

$$L := \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial x^2} \quad \text{on } (0, \infty) \times \mathbb{R}. \quad (6.4)$$

That is,

$$p_t(x) := \frac{e^{-x^2/(2t)}}{\sqrt{2\pi t}}, \quad (6.5)$$

for every $t > 0$ and $x \in \mathbb{R}$.

It is also well known—see Walsh [*ibid.*]—that the random field Z has a modification that is continuous in (t, x) ; from now on we always use that version of the random field Z in order to avoid measurability issues.

We are interested in the structure of the tall peaks of the random field $x \mapsto Z_t(x)$, where $t > 0$ is held fixed. With this aim in mind, consider the random set

$$\mathcal{P}_{Z_t}(\gamma) := \left\{ x \geq \exp(e) : \frac{Z_t(x)}{(2 \log x)^{1/2}} \geq \gamma \left(\frac{t}{\pi} \right)^{1/4} \right\}, \quad (6.6)$$

where $t, \gamma > 0$ are fixed.

Theorem 6.1. *Every $\mathcal{P}_{Z_t}(\gamma)$ is almost surely unbounded if $\gamma \leq 1$; else, if $\gamma > 1$ then $\mathcal{P}_{Z_t}(\gamma)$ is almost surely bounded. Furthermore,*

$$\text{Dim}_{\mathbb{H}} \mathcal{P}_{Z_t}(\gamma) = 1 - \gamma^2 \quad \text{a.s.}, \quad (6.7)$$

for all $t > 0$ and $\gamma \in (0, 1]$.

A moment's thought shows that Theorem 6.1 is an equivalent reformulation of Theorem 1.3 of Introduction. From a technical point of view, this particular formulation is more natural to state as well as prove.

As we will see very soon, the fact that $\gamma = 1$ is critical for the unboundedness of $\mathcal{P}_{Z_t}(\gamma)$ is a fairly routine consequence of well-known theorems about the growth of a Gaussian process [37]. The main assertion of Theorem 6.1 is the one about the macroscopic Hausdorff dimension of $\mathcal{P}_{Z_t}(\gamma)$. Still, let us mention also the following immediate consequence of the first [more or less routine] portion of Theorem 6.1:

$$\limsup_{x \rightarrow \infty} \frac{Z_t(x)}{(2 \log x)^{1/2}} = \left(\frac{t}{\pi}\right)^{1/4} \quad \text{a.s.}, \quad (6.8)$$

for every nonrandom $t > 0$. A “steady state” version of this fact appears earlier, for example, in Collela and Lanford [10, Theorem 1.1(c)]. The following lemma puts things in the general framework of Gaussian analysis.

Lemma 6.2. *Fix some $t > 0$. Then, $\{Z_t(x)\}_{x \in \mathbb{R}}$ is a stationary Gaussian process with $\mathbb{E}Z_t(0) = 0$, $\text{Var } Z_t(0) = (t/\pi)^{1/2}$, and $\text{Corr}[Z_t(x), Z_t(0)] = O(|x|^{-a})$ as $|x| \rightarrow \infty$ for every $a > 0$. Finally,*

$$\text{Corr}[Z_t(x), Z_t(0)] = 1 - \frac{1}{2} \left(\frac{\pi}{t}\right)^{1/2} |x| + o(|x|) \quad \text{as } |x| \rightarrow 0. \quad (6.9)$$

Proof. Clearly, $x \mapsto Z_t(x)$ is a mean-zero Gaussian process with

$$\text{Cov}[Z_t(x), Z_t(x')] = \int_0^t p_{2s}(x - x') \, ds \quad \text{for all } x, x' \in \mathbb{R}. \quad (6.10)$$

It follows from this formula that $Z_t(\bullet)$ is stationary as well, and has variance

$$\text{Var } Z_t(0) = \int_0^t p_{2s}(0) \, ds = \int_0^t (4\pi s)^{-1/2} \, ds = \left(\frac{t}{\pi}\right)^{1/2}. \quad (6.11)$$

Furthermore, the preceding display shows also that

$$\text{Cov}[Z_t(x), Z_t(0)] = \int_0^t p_{2s}(x) \, ds \quad (6.12)$$

is bounded above by a finite constant $C(t)$ times $e^{-x^2/(4t)}$, and hence goes to zero faster than any negative power of $|x|$, as $|x| \rightarrow \infty$. Finally, we note that if $x > 0$, then

$$\begin{aligned} \text{Var}(Z_t(0)) - \text{Cov}[Z_t(x), Z_t(0)] &= \int_0^t [p_{2s}(0) - p_{2s}(x)] \, ds \\ &= \frac{x}{4\sqrt{\pi}} \int_0^{4t/x^2} r^{-1/2} (1 - e^{-1/r}) \, dr \\ &= \frac{x}{4\sqrt{\pi}} \int_0^\infty r^{-1/2} (1 - e^{-1/r}) \, dr + O(x^2), \end{aligned} \quad (6.13)$$

as $x \downarrow 0$. It follows readily from this and symmetry that

$$\text{Corr}[Z_t(x), Z_t(0)] = 1 - c|x| + O(x^2) \quad \text{as } x \rightarrow 0, \quad (6.14)$$

with

$$c := \frac{1}{4\sqrt{t}} \int_0^\infty r^{-1/2} (1 - e^{-1/r}) dr. \quad (6.15)$$

A change of variables shows that

$$\int_0^\infty r^{-1/2} (1 - e^{-1/r}) dr = \int_0^\infty s^{-3/2} (1 - e^{-s}) ds. \quad (6.16)$$

Write $1 - e^{-s} = \int_0^s e^{-y} dy$ and apply the Tonelli theorem in order to see that $c = \sqrt{\pi/4t}$. \square

We are ready to establish Theorem 6.1.

Proof of Theorem 6.1. Throughout the proof, we hold fixed an arbitrary $t > 0$.

Lemma 6.2 verifies all of the conditions of Theorem 1.1 of Qualls and Watanabe [37], and hence it follows from that result that $\mathcal{P}_{Z_t}(\gamma)$ is a.s. bounded if $\gamma > 1$ and a.s. unbounded if $\gamma \leq 1$. In particular, we obtain (6.8) immediately. Furthermore, we can see—using the notation of §4—that

$$b = 2 \text{ and } c(b) = C(b) = \left(\frac{\pi}{4t}\right)^{1/2}. \quad (6.17)$$

Thanks to Lemma 6.2, the assumptions of Lemma 2.5 of Pickands [36] are met. Lemma 4.9 [Pickands' theorem] implies the maximal inequality (4.5); therefore, we may apply Theorem 4.1—with γ replaced by γ^2 —in order to conclude that

$$\text{Dim}_{\text{H}} \mathcal{P}_{Z_t}(\gamma) \leq 1 - \gamma^2, \quad (6.18)$$

a.s. for all $\gamma \in (0, 1]$. We plan to prove a matching lower bound by appealing to Theorem 4.7 with $S(x) := x$ for all $x \in \mathbb{R}$. Therefore, it remains to verify the coupling assumption (4.34), which we do next.

For every $B > 0$ we may define a space-time Gaussian random field $Z^{(B)}$ as follows: For all $x \in \mathbb{R}$,

$$Z_t^{(B)}(x) := \int_{(0,t) \times [x - (Bt)^{1/2}, x + (Bt)^{1/2}]} p_{t-s}(y - x) \xi(ds dy). \quad (6.19)$$

It is intuitively clear that $Z \approx Z^{(B)}$ when $B \gg 1$. Next we claim the following quantitative improvement of this remark: For all $t, B, \lambda > 0$,

$$\sup_{x \in \mathbb{R}} \mathbb{P} \left\{ \left| Z_t(x) - Z_t^{(B)}(x) \right| > \lambda \right\} \leq 2 \exp \left(-\frac{\lambda^2}{2} \sqrt{\frac{\pi}{8t}} e^{B/2} \right). \quad (6.20)$$

Indeed, because $p_s(z) \leq p_s(0) = (2\pi s)^{-1/2}$ for all $s > 0$ and $z \in \mathbb{R}$, the Wiener isometry yields

$$\begin{aligned} \mathbb{E} \left(\left| Z_t(x) - Z_t^{(B)}(x) \right|^2 \right) &= \int_0^t ds \int_{|z| > (Bs)^{1/2}} dz [p_s(z)]^2 \\ &\leq \int_0^t \frac{ds}{\sqrt{2\pi s}} \mathbb{P} \left\{ |X| > (Bs/s)^{1/2} \right\}, \end{aligned} \quad (6.21)$$

where X has a standard normal distribution. If $s \in (0, t)$, then we combine the elementary bound,

$$\mathbb{P} \left\{ |X| > \left(\frac{Bs}{s} \right)^{1/2} \right\} \leq \mathbb{P} \left\{ |X| > \sqrt{B} \right\}, \quad (6.22)$$

with a standard bound on the tails of the standard normal distribution in order to see

$$\text{Var} \left(Z_t(x) - Z_t^{(B)}(x) \right) \leq \left(\frac{8t}{\pi} \right)^{1/2} e^{-B/2}. \quad (6.23)$$

The claim (6.20) follows readily from this and another appeal to the tails of the Gaussian laws.

We use (6.20) in order to prove (4.34) using the following.

Observation 1. *If $x_1, x_2, \dots, x_m \in \mathbb{R}$ satisfy $|x_i - x_j| > 2(Bt)^{1/2}$ when $1 \leq i \neq j \leq m$, then the random variables $Z_t^{(B)}(x_1), \dots, Z_t^{(B)}(x_m)$ are independent.*

Choose and fix some $\delta \in (0, 1)$. If $e^n \leq x_1 < \dots < x_m < e^{n+1}$ are m arbitrary points in \mathcal{S}_n such that $x_{i+1} - x_i \geq \exp\{\delta n\}$, then we set $Y_j := Z_t^{(n)}(x_j)$ for all $1 \leq j \leq m$. Thanks to Observation 1, Y_1, \dots, Y_m are independent random variables as long as n is large enough to ensure that $2\sqrt{nt} < \exp\{\delta n\}$. And (6.20) ensures that

$$\max_{1 \leq i \leq m} \mathbb{P} \{ |Z_t(x_i) - Y_i| > 1 \} \leq 2 \exp \left(-\frac{1}{2} \sqrt{\frac{\pi}{8t}} e^{n/2} \right). \quad (6.24)$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \max_{1 \leq i \leq m} \log \mathbb{P} \{ |Z_t(x_i) - Y_i| > 1 \} = -\infty. \quad (6.25)$$

This implies (4.34) readily, and the lower bound that complements (6.18) follows from the conclusion of Theorem 4.7. \square

7 Peaks of a nonlinear stochastic heat equation

Let us now consider the following nonlinear stochastic partial differential equation,

$$\dot{u}_t(x) = \frac{1}{2} u_t''(x) + \sigma(u_t(x)) \xi_t(x) \quad (x \in \mathbb{R}, t > 0), \quad (7.1)$$

$$u_0(x) = 1, \quad (7.2)$$

where ξ denotes space-time white noise, as before, and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous and non-random function with $\sigma(0) = 0$.

It is well-known that the stochastic heat equation (7.1) has a unique solution; see Dalang [16], for instance. And that solution solves the following stochastic integral equation, interpreted in the sense of Walsh [39]:

$$u_t(x) = 1 + \int_{(0,t) \times \mathbb{R}} p_{t-s}(y-x) \sigma(u_s(y)) \xi(ds dy), \quad (7.3)$$

where $p_t(x)$ is the standard heat kernel on $(0, \infty) \times \mathbb{R}$; see (6.5).

It is known also that the solution to (7.1) is strictly positive for all $t > 0$; see Mueller [30] for a closely-related statement. The precise positivity assertion that is required here follows from the work of Mueller and Nualart [31]. Therefore, the tall peaks of $x \mapsto u_t(x)$ and $x \mapsto h_t(x)$ match, where

$$h_t(x) := \log u_t(x). \quad (7.4)$$

The random field h is particularly well studied when $\sigma(z) = z$ for all $z \in \mathbb{R}$. In that case, a formal change of variables suggests that

$$\dot{h}_t(x) = \frac{1}{2} h_t''(x) + \frac{1}{2} (h_t'(x))^2 - \xi_t(x), \quad (7.5)$$

subject to $h_0(x) \equiv 0$. This purely-formal “computation” is analogous to the classical Hopf–Cole solution to Burgers’ equation, and is due to Kardar et al [26]. The resulting ill-posed stochastic PDE (7.5) is the so-called “KPZ equation” of statistical mechanics. The recent solution theory of Hairer [21] gives a meaning to the analogous version of (7.5) where the x variable lives in $[0, 1]$ [together with suitable boundary conditions]. As far as we know, the original problem on \mathbb{R} has not yet been given a rigorous meaning.

In this section, we plan to study the set of points $x > \exp(e)$ at which the solution $u_t(x)$ exceeds certain high peaks. For the parabolic Anderson model—that is when $\sigma(z) = z$ for all $z \in \mathbb{R}$ —Conus et al [12] have demonstrated that, for every $t > 0$, the tall peaks of $x \mapsto h_t(x)$ are of rough height $(\log |x|)^{2/3}$ as $|x| \rightarrow \infty$. Specifically, they have proved that

$$0 < \limsup_{x \rightarrow \infty} \frac{h_t(x)}{(\log x)^{2/3}} < \infty \quad \text{a.s.}, \quad (7.6)$$

for all $t > 0$. Conus et al [*ibid.*] have also proved that the function $(\log x)^{2/3}$ fails to correctly gauge the height of the tall peaks of $h_t(x)$ for general nonlinearities σ .

We will prove among other things that (7.6) holds when $|\sigma(z)/z|$ is bounded uniformly from below by a positive constant. The mentioned boundedness condition is known to be an *intermittency condition* for the system (7.1) [18].

In order to describe our results in greater details let us define

$$\ell_\sigma := \inf_{z \in \mathbb{R} \setminus \{0\}} |\sigma(z)/z|, \quad L_\sigma := \sup_{z \in \mathbb{R} \setminus \{0\}} |\sigma(z)/z|. \quad (7.7)$$

Because σ is Lipschitz continuous we always have $0 \leq \ell_\sigma \leq L_\sigma \leq \infty$. We will be assuming that

$$0 < \ell_\sigma \leq L_\sigma < \infty. \quad (7.8)$$

We call (7.8) an “intermittency condition” because it is the only known condition under which the solution to (7.1) is known to be intermittent in the sense that

$$k \mapsto \frac{\lambda(k)}{k} \text{ is strictly increasing on } [2, \infty), \quad (7.9)$$

where either

$$\lambda(k) := \limsup_{t \rightarrow \infty} t^{-1} \sup_{x \in \mathbb{R}} \mathbb{E} (|u_t(x)|^k), \quad (7.10)$$

or

$$\lambda(k) := \liminf_{t \rightarrow \infty} t^{-1} \inf_{x \in \mathbb{R}} \mathbb{E} (|u_t(x)|^k), \quad (7.11)$$

describe respectively the top and bottom k th moment Lyapunov exponents of the solution; see Foondun and Khoshnevisan [18, Theorem 2.7].⁵

Now define for all $t, \gamma > 0$,

$$\mathcal{P}_{h_t}(\gamma) := \left\{ x \geq \exp(e) : \frac{h_t(x)}{(\log x)^{2/3}} \geq \gamma t^{1/3} \right\}. \quad (7.12)$$

We will use the general theory of §4 in order to prove the following, which is the main result of this section. It might help to recall yet again our earlier convention that when we state that $\text{Dim}_{\mathbb{H}} E < 0$ we mean that E is bounded.

Theorem 7.1. *Under (7.8), the following holds with probability one:*

$$\left(\frac{9}{32} \right)^{1/3} \ell_\sigma^{4/3} \leq \limsup_{x \rightarrow \infty} \frac{h_t(x)}{t^{1/3} (\log x)^{2/3}} \leq \left(\frac{9}{32} \right)^{1/3} L_\sigma^{4/3}; \quad (7.13)$$

for every $t, \gamma > 0$. Moreover,

$$1 - \alpha \gamma^{3/2} \leq \text{Dim}_{\mathbb{H}} \mathcal{P}_{h_t}(\gamma) \leq 1 - \beta \gamma^{3/2} \quad a.s., \quad (7.14)$$

where $\mathcal{P}_{h_t}(\gamma)$ was defined in (7.12),

$$\alpha := \frac{4\sqrt{2}}{3\ell_\sigma^2}, \quad \text{and} \quad \beta := \frac{4\sqrt{2}}{3L_\sigma^2}. \quad (7.15)$$

When $\sigma(z) = z$ for all $z \in \mathbb{R}$, the stochastic PDE (7.1) simplifies to the following, which is known as a *parabolic Anderson model* and/or *diffusion in random white-noise potential*:

$$\begin{cases} \dot{u}_t(x) = \frac{1}{2} u_t''(x) + u_t(x) \xi_t(x) & (x \in \mathbb{R}, t > 0); \\ u_0(x) = 1. \end{cases} \quad (7.16)$$

In this case, Theorem 7.1 yields the following exact formula, which is an equivalent but perhaps more explicit way to state Theorem 1.2.

⁵In general, the Lyapunov exponents, as were describe in the introduction, do not exists.

Corollary 7.2. *The solution u to (7.16) satisfies the following: For all $\gamma, t > 0$,*

$$\text{Dim}_H \left\{ x \geq \exp(e) : u_t(x) \geq e^{\gamma t^{1/3} (\log x)^{2/3}} \right\} = 1 - \frac{4\sqrt{2}}{3} \gamma^{3/2} \quad a.s. \quad (7.17)$$

Let us mention a rather general corollary of Theorem 7.1 as well.

Corollary 7.3. *Under (7.8), the tall peaks of the solution to the SPDE (7.1) are almost surely multifractal.*

Proof. Recall (7.15), let $\gamma_1 := (2\alpha)^{-2/3}$, and then define

$$\gamma_{i+1} := \left(\frac{\beta}{2\alpha} \right)^{2/3} \gamma_i, \quad (7.18)$$

iteratively for all $i \geq 1$. Clearly,

$$1 - \beta \gamma_i^{3/2} < 1 - \frac{1}{2} \beta \gamma_i^{3/2} = 1 - \alpha \gamma_{i+1}^{3/2} \quad \text{for all } i \geq 1. \quad (7.19)$$

In addition, $0 < 1 - \alpha \gamma_i^{3/2} < 1 - \beta \gamma_i^{3/2} < 1$ for all $i \geq 1$.

Consider the collection of tall peaks $\mathcal{P}_{h_t}(\gamma)$ of order $\gamma \in (0, 1)$ that was defined in (7.12). Theorem 7.1 implies that

$$\text{Dim}_H \mathcal{P}_{h_t}(\gamma_i) < \text{Dim}_H \mathcal{P}_{h_t}(\gamma_{i+1}) \quad \text{for all } i \geq 1, \text{ a.s.} \quad (7.20)$$

Definition 1.1 then shows that the tall peaks of h_t —whence also u_t —are a.s. multifractal. \square

We begin the proof of Theorem 7.1 with a basic tail probability estimate.

Proposition 7.4. *For any $t > 0$, we have*

$$\begin{aligned} \liminf_{z \rightarrow \infty} z^{-3/2} \inf_{x \in \mathbb{R}} \log \mathbb{P} \{ h_t(x) \geq z \} &\geq -\frac{\alpha}{\sqrt{t}}, \\ \limsup_{z \rightarrow \infty} z^{-3/2} \sup_{x \in \mathbb{R}} \log \mathbb{P} \{ h_t(x) \geq z \} &\leq -\frac{\beta}{\sqrt{t}}, \end{aligned} \quad (7.21)$$

where the constants α and β were defined in (7.15).

Proof. Let $u^{(\ell)}$ and $u^{(L)}$ respectively denote the solutions to (7.1) with $\sigma(z) := \ell_\sigma z$ and $\sigma(z) := L_\sigma z$. The *moment comparison principle* of Joseph et al [24, Theorem 2.6] tells us that because of the condition (7.8),

$$\mathbb{E} \left(\left[u_t^{(\ell)}(x) \right]^k \right) \leq \mathbb{E} \left([u_t(x)]^k \right) \leq \mathbb{E} \left(\left[u_t^{(L)}(x) \right]^k \right), \quad (7.22)$$

for all real numbers $t > 0$, $x \in \mathbb{R}$, and $k \geq 2$. We now use the first part of Theorem 5.5 of Chen [8] and the Gärtner–Ellis theorem, for example in the form of Chapter 1 of the recent book by Chen [9], in order to complete the proof. \square

Armed with Proposition 7.4 we can prove half of Theorem 7.1 quickly. The second, harder, half will require work that will be developed afterward.

Proof of Theorem 7.1: Dimension upper bound. Our immediate goal is to establish the dimension upper bound; that is, we wish to demonstrate the following:

$$\text{Dim}_H \mathcal{P}_{h_t} \left([\gamma/\beta]^{2/3} \right) \leq 1 - \gamma \quad \text{a.s. for all } \gamma > 0. \quad (7.23)$$

We claim that for every $\gamma \in (0, 1)$,

$$\sup_{z \in \mathbb{R}} \mathbb{P} \left\{ \sup_{y \in [z, z+1]} h_t(y) > \left(\frac{\gamma}{\beta} \log s \right)^{2/3} \right\} \leq s^{-\gamma+o(1)} \quad \text{as } s \rightarrow \infty. \quad (7.24)$$

If this were so, then it would show that Condition (4.5) holds with $b = 3/2$ and $X_x := h_t(x)$, and Theorem 4.1 then implies (7.23). Thus, the dimension upper bound of Theorem 7.1 follows once we prove (7.24). According to (7.4), it remains to prove that for every $\gamma \in (0, 1)$,

$$\sup_{z \in \mathbb{R}} \mathbb{P} \left\{ \sup_{y \in [z, z+1]} u_t(y) > \exp \left[\left(\frac{\gamma}{\beta} \log s \right)^{2/3} \right] \right\} \leq s^{-\gamma+o(1)}, \quad (7.25)$$

as $s \rightarrow \infty$.

Recall that, as a corollary to Proposition 7.4, we have the following slightly weaker variation on the desired estimate (7.25): If $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nonrandom function that satisfies $\lim_{s \rightarrow \infty} g(s) = 0$, then

$$\sup_{y \in \mathbb{R}} \mathbb{P} \left\{ u_t(y) > \exp \left[\left(\frac{\gamma - g(s)}{\beta} \log s \right)^{2/3} \right] \right\} \leq s^{-\gamma+o(1)}, \quad (7.26)$$

as $s \rightarrow \infty$.

In order to derive (7.25) from (7.26) we apply a chaining argument. With this in mind, let us first observe the following, which is a quantitative form of the Kolmogorov continuity theorem [28, Theorem C.6, p. 107]: There exists a finite constant $\tau = \tau(t) > 1$ such that for all real numbers $k \geq 2$

$$\sup_{w \in \mathbb{R}} \mathbb{E} \left[\sup_{\substack{x, x' \in [w, w+1] \\ x \neq x'}} \frac{|u_t(x) - u_t(x')|^{2k}}{|x - x'|^{k/2}} \right] < \tau e^{\tau k^3 t}. \quad (7.27)$$

This and the Chebyshev inequality together imply that uniformly for all real numbers $\eta, \varepsilon \in (0, 1)$, $s \geq e$, and $k \geq 2$,

$$\begin{aligned} \sup_{w \in \mathbb{R}} \mathbb{P} \left\{ \sup_{y \in [w, w+\varepsilon]} |u_t(w) - u_t(y)| > \exp \left[(\eta \log s)^{2/3} \right] \right\} \\ \leq \tau \varepsilon^{k/2} \exp \left(\tau k^3 t - 2k(\eta \log s)^{2/3} \right). \end{aligned} \quad (7.28)$$

We apply the preceding bound with the following choices of ε and k :

$$\begin{aligned}\varepsilon &= \varepsilon(s) := \exp \left\{ -\frac{2\gamma(2\tau t)^{1/2}}{\eta^{1/3}} (\log s)^{2/3} \right\}, \\ k &= k(s) := \left(\frac{2}{\tau t} \right)^{1/2} (\eta \log s)^{1/3}.\end{aligned}\tag{7.29}$$

Let $s_*(\eta, t) := \exp(\eta^{-1}[2\tau t]^{3/2})$ to see that $k \geq 2$ if and only if $s \geq s_*(\eta, t)$. We apply (7.28) with these choices of ε and k in order to see that

$$\sup_{w \in \mathbb{R}} \mathbb{P} \left\{ \sup_{y \in [w, w+\varepsilon]} |u_t(w) - u_t(y)| > \exp \left[(\eta \log s)^{2/3} \right] \right\} \leq \tau s^{-2\gamma}, \tag{7.30}$$

uniformly for all $\eta \in (0, 1)$ and $s \geq \max\{e, s_*(\eta, t)\}$. In particular, we might note that

$$\begin{aligned}& \sup_{w \in \mathbb{R}} \mathbb{P} \left\{ \sup_{y \in [w, w+\varepsilon]} u_t(y) > \exp \left[\left(\frac{\gamma}{\beta} \log s \right)^{2/3} \right] \right\} \\ & \leq \tau s^{-2\gamma} + \sup_{w \in \mathbb{R}} \mathbb{P} \left\{ u_t(w) > \exp \left[\left(\frac{\gamma}{\beta} \log s \right)^{2/3} \right] - \exp \left[(\eta \log s)^{2/3} \right] \right\} \\ & \leq s^{-\gamma+o(1)} \quad \text{as } s \rightarrow \infty,\end{aligned}\tag{7.31}$$

owing to (7.26). Every interval $[z, z+1]$ can be covered by at most $\varepsilon^{-1} + 1$ intervals of the form $[w, w+\varepsilon]$. Therefore, the preceding implies that

$$\sup_{z \in \mathbb{R}} \mathbb{P} \left\{ \sup_{y \in [z, z+1]} u_t(y) > \exp \left[\left(\frac{\gamma}{\beta} \log s \right)^{2/3} \right] \right\} \leq [\varepsilon^{-1} + 1] s^{-\gamma+o(1)} = s^{-\gamma+o(1)},$$

as $s \rightarrow \infty$. This proves (7.25), and hence the upper bound on the macroscopic Hausdorff dimension in Theorem 7.1. \square

Now we begin to work toward establishing the lower bound in Theorem 7.1. In order to do that we will attempt to verify the coupling condition (4.34). A first attempt might be to follow the case of linear SPDEs/Gaussian processes. More concretely, we may try to follow the proof of the lower bound of Theorem 6.1 and consider, for every $B > 0$, a space-time random field $u^{(B)}$ as follows: For all $t > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}u_t^{(B)}(x) \\ := 1 + \int_{(0,t) \times [x-(Bt)^{1/2}, x+(Bt)^{1/2}]} p_{t-s}(y-x) \sigma \left(u_s^{(B)}(y) \right) \xi(ds dy).\end{aligned}\tag{7.32}$$

It is not hard to apply a fixed-point argument in order to prove that the random integral equation (7.32) has a unique solution $u^{(B)}$. Moreover, that $(t, x) \mapsto$

$u_t^{(B)}(x)$ has a continuous modification. [We will not prove any of this here since we will not need to.]

The random field $u^{(B)}$ is the analogue of the random field $Z^{(B)}$, that was defined earlier in (6.19), but we now interpret the stochastic integral in (7.32) in the sense of Walsh, whereas the one for $Z^{(B)}$ can be understood in the sense of Wiener.

The random field $Z^{(B)}$ was introduced in the proof of Theorem 6.1 because $Z^{(B)}$ has the following two desirable properties:

- (i) $Z^{(B)} \approx Z$ if B is large [see (6.20)]; and
- (ii) $Z_t^{(B)}(x_1), \dots, Z_t^{(B)}(x_m)$ are independent if the x_i 's are sufficiently far apart from one another; for example, if $|x_i - x_{i+1}| > 2(Bt)^{1/2}$ [see Observation 1.]

By analogy, we might hope that:

- (iii) $u^{(B)} \approx u$ if B is large; and
- (iv) $u_t^{(B)}(x_1), \dots, u_t^{(B)}(x_m)$ are independent if the x_i 's are sufficiently far apart from one another.

If so, then we could use $u^{(B)}$ —in a similar way as we used $Z^{(B)}$ —in order to verify Condition (4.34), thereby obtain a lower bound on the macroscopic dimension of the set of high peaks of u .

As it turns out, (iii) continues to hold. However, (iv) is manifestly false; it is possible for example to show that the covariance of $u_t^{(B)}(x_1)$ and $u_t^{(B)}(x_2)$ is strictly positive, for all $x_1, x_2 \in \mathbb{R}$, no matter how far apart x_1 and x_2 are from one another.

We remedy the situation by defining the following random fields instead: Choose and fix an integer $B \geq 1$, as before, and define

$$u_t^{(B,0)}(x) := 1 \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}. \quad (7.33)$$

Then we define random fields $u^{(B,j)}$, for every $j \geq 1$, iteratively, as follows:

$$u_t^{(B,m)}(x) := 1 + \int_{(0,t) \times [x-(Bt)^{1/2}, x+(Bt)^{1/2}]} p_{t-s}(y-x) \sigma \left(u_s^{(B,m-1)}(y) \right) \xi(ds dy),$$

for every $m \geq 1$. The object of interest to us is the random field $u^{(B,B)}$. The following estimate shows that $u \approx u^{(B,B)}$ when $B \gg 1$.

Lemma 7.5 (A coupling lemma). *There exists a finite constant $K > 1$ such that for all real numbers $t > 0$ and $\lambda > 1$, and all integers $B \geq 1$,*

$$\sup_{x \in \mathbb{R}} \mathbb{P} \left\{ \left| u_t(x) - u_t^{(B,B)}(x) \right| > \lambda \right\} \leq K \exp \left(- \frac{(B + \log \lambda)^{3/2}}{K \sqrt{t}} \right). \quad (7.34)$$

Proof. According to Lemma 4.3 of Conus et al [12]—see also Lemma 10.10 of Khoshnevisan [28]—there exists a finite constant c such that

$$\sup_{x \in \mathbb{R}} \mathbb{E} \left(\left| u_t(x) - u_t^{(B,B)}(x) \right|^k \right) \leq c \exp(ck^3t - Bk), \quad (7.35)$$

uniformly for all real numbers $k \geq 2$, $\lambda > 1$, and $t > 0$. Therefore, Chebyshev's inequality shows that

$$\mathbb{P} \left\{ \left| u_t(x) - u_t^{(B,B)}(x) \right| > \lambda \right\} \leq c \inf_{k \geq 2} \exp(ck^3t - (\log \lambda + B)k), \quad (7.36)$$

uniformly for all real numbers $t > 0$ and $x \in \mathbb{R}$, and for all integers $B \geq 1$. This readily implies the lemma. \square

Our next lemma is essentially due to Conus et al [12], and shows that the random process $x \mapsto u_t^{(B,B)}(x)$ decouples fairly rapidly.

Lemma 7.6. *Suppose $t > 0$ is a real number, $B \geq 1$ is an integer, and x_1, \dots, x_m are points in \mathbb{R} such that $|x_i - x_j| > 2B^{3/2}\sqrt{t}$ whenever $i \neq j$. Then, $u_t^{(B,B)}(x_1), \dots, u_t^{(B,B)}(x_m)$ are independent random variables.*

Proof. We include a proof for the sake of completeness.

First, let us observe that if $|x_i - x_j| > 2(Bt)^{1/2}$, whenever $i \neq j$, then $u_t^{(B,1)}(x_1), \dots, u_t^{(B,1)}(x_m)$ are independent. This is because: (i)

$$u_t^{(B,1)}(x) = 1 + \int_{(0,t) \times [x-(Bt)^{1/2}, x+(Bt)^{1/2}]} p_{t-s}(y-x) \sigma(1) \xi(ds dy); \quad (7.37)$$

and (ii) If $\psi_1, \dots, \psi_m \in L^2(\mathbb{R}_+ \times \mathbb{R})$ are nonrandom with disjoint support then the Wiener integrals $\int \psi_j d\xi$ ($1 \leq j \leq m$) are independent [compute covariances]. Next we apply induction, using the following induction hypothesis: Suppose that whenever $|x_i - x_j| > 2\ell(Bt)^{1/2}$ for $i \neq j$, the random variables $u_t^{(B,\ell)}(x_1), \dots, u_t^{(B,\ell)}(x_m)$ are independent. Then we wish to prove that if $|x_i - x_j| > 2(\ell+1)(Bt)^{1/2}$ [$i \neq j$], then $u_t^{(B,\ell+1)}(x_1), \dots, u_t^{(B,\ell+1)}(x_m)$ are independent. This property follows readily from the properties of the Walsh stochastic integral; namely, that if Φ^1, \dots, Φ^m are independent predictable random fields and $\psi^1, \dots, \psi^m \in L^2(\mathbb{R}_+ \times \mathbb{R})$ are nonrandom with disjoint support, then the Walsh integrals $\int \psi_j \Phi^j d\xi$ are independent [$1 \leq j \leq m$]. This completes our induction argument, and prove the lemma. \square

We can now verify the dimension lower bound of Theorem 7.1.

Proof of Theorem 7.1: Dimension lower bound. Our proof of Theorem 7.1 will be complete once we demonstrate that

$$\text{Dim}_{\mathbb{H}} \mathcal{P}_{h_t} \left([\gamma/\alpha]^{2/3} \right) \geq 1 - \gamma \quad \text{a.s. for all } \gamma > 0. \quad (7.38)$$

In order to establish this fact we will appeal to Theorem 4.7; therefore, it remains to verify Condition (4.34). We will appeal, as we did in the proof of the upper bound, to the general theory of Section 4, using the identifications $X_x := h_t(x)$, $b := 3/2$, and $S(x) := \exp(x)$.

Let us choose and fix an integer $n \geq 1$ and a real number $\delta \in (0, 1)$, and consider an arbitrary collection $\{x_i\}_{i=1}^m$ of points such that: (a) $e^n \leq x_1 < \dots < x_m < e^{n+1}$; and (b) $x_{i+1} - x_i \geq \exp(\delta n)$. From now on, we set $B := n^3$ and

$$Y_j := \log u_t^{(B,B)}(x_j) \quad (1 \leq j \leq m). \quad (7.39)$$

According to Lemma 7.6, Y_1, \dots, Y_m are independent as long as $2n^3 t < \exp(\delta n)$; and Lemma 7.5 ensures that

$$\max_{1 \leq j \leq m} \mathbb{P} \{ |S(X_{x_j}) - S(Y_j)| > 1 \} \leq K \exp \left(-\frac{n^{3/2}}{K\sqrt{t}} \right). \quad (7.40)$$

[Recall that $S(x) := x$ here.] Since the constant K does not depend on the choice of $x_1, \dots, x_m \in \mathcal{S}_n$, we have shown that Condition (4.34) holds. Theorem 4.7 implies (7.38), and completes our proof of Theorem 7.1. \square

8 d -dimensional diffusion in random potential

We will conclude this paper by presenting examples of stochastic PDEs over $\mathbb{R}_+ \times \mathbb{R}^d$ where $d \geq 1$ is an arbitrary positive integer. In order to keep the ensuing theory at a reasonably-modest technical level, we focus only on linear stochastic partial differential equations of the following type:

$$\begin{cases} \dot{u}_t(x) = \frac{1}{2}(\Delta u_t)(x) + u_t(x)\eta_t(x) & \text{for } t > 0, x \in \mathbb{R}^d; \\ u_0(x) = 1; \end{cases} \quad (8.1)$$

where the Laplace operator Δ acts on the variable $x \in \mathbb{R}^d$, and $\eta := \{\eta_t(x)\}_{t \geq 0, x \in \mathbb{R}^d}$ is a centered generalized Gaussian random field with covariance measure

$$\text{Cov}[\eta_t(x), \eta_s(y)] = \delta_0(t - s)f(x - y) \quad [s, t \geq 0, x, y \in \mathbb{R}^d], \quad (8.2)$$

for a positive-definite bounded and continuous function $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$.

By comparison to the classical situation of Brownian heat baths, we can think of the solution to (8.1) as describing Brownian motion in a random environment.

When $d = 3$, a variation of equations of the form (8.1) was introduced in cosmology in order to describe the large-scale structure of the universe. Here are some more details: It is believed that,⁶ after we make a standard change of

⁶See Albeverio et al [1] and its detailed references to the physics literature, in particular, to the pioneering work of Zel'dovich and his collaborators.

variables to remove mathematically-uninteresting physical constants, the velocity field \vec{v} of galaxy masses approximately solves the 3-dimensional stochastic PDE,⁷

$$\frac{\partial}{\partial t} \vec{v}_t(x) = \frac{1}{2}(\Delta \vec{v}_t)(x) - (\vec{v}_t(x))^\top \nabla \vec{v}_t(x) + \nabla \Phi_t(x), \quad (8.3)$$

for $(t, x) \in (0, \infty) \times \mathbb{R}^3$, subject to the following:

$$\text{curl}(\vec{v}_t(x)) = 0, \quad \vec{v}_0(x) = -\nabla \psi(x); \quad (8.4)$$

for all $t > 0$ and $x \in \mathbb{R}^3$. The external field Φ is a scalar field and believed to be random, and the initial field ψ may or may not be random. Now we apply a formal Hopf–Cole transform and posit that

$$\vec{v}_t(x) = -\nabla \log \phi_t(x), \quad (8.5)$$

for a scalar field ϕ . It is then easy to see that, if ψ and Φ were smooth, then ϕ would solve

$$\begin{cases} \dot{\phi}_t(x) = \frac{1}{2}(\Delta \phi_t)(x) + \phi_t(x)\Phi_t(x) & \text{for } t > 0, x \in \mathbb{R}^3; \\ \phi_0(x) = e^{\psi(x)}. \end{cases} \quad (8.6)$$

As far as we know, there is no general agreement on what the external field Φ should be, though the simplest form of the “big-bang theory” might suggest that $\psi = \delta_0$, after a suitable relabeling of \mathbb{R}^3 . Our stochastic PDE (8.1) is this equation in the particular case that $\Phi \equiv \eta$ and $\psi \equiv 0$.

8.1 The main result

It is a classical fact that our correlation function f is uniformly continuous and maximized at the origin; i.e.,

$$f(z) \leq f(0) \quad \text{for all } z \in \mathbb{R}^d. \quad (8.7)$$

Therefore, in order to avoid trivialities we will always assume that

$$f(0) > 0. \quad (8.8)$$

Indeed, if $f(0)$ were zero, then $f \equiv 0$ and hence $\eta \equiv 0$. In that case, the solution to (8.1) is $u_t(x) \equiv 1$, trivially.

Let \hat{f} denote the distributional Fourier transform of f . Because f is assumed to be positive definite, \hat{f} is a positive distribution. That is, \hat{f} is a tempered Borel measure on \mathbb{R}^d thanks to the Riesz representation theorem. Furthermore, the Parseval identity shows that

$$\int_{\mathbb{R}^d} |\hat{\varphi}(x)|^2 \hat{f}(dx) = (2\pi)^d \int_{\mathbb{R}^d} (\varphi * \tilde{\varphi})(z) f(z) dz \leq (2\pi)^d f(0) \|\varphi\|_{L^1(\mathbb{R}^d)}^2, \quad (8.9)$$

⁷Here, $\nabla \vec{v}$ denotes the matrix of all first derivatives of the coordinate functions of \vec{v} .

for all rapidly-decreasing test functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, where $\tilde{\varphi}$ denotes the *reflection* of φ ; that is,

$$\tilde{\varphi}(x) := \varphi(-x) \quad \text{for all } x \in \mathbb{R}^d. \quad (8.10)$$

We replace φ by φ_ϵ —where $\{\varphi_\epsilon\}_{\epsilon>0}$ is an approximate identity built from functions in $\mathcal{S}(\mathbb{R}^d)$ —and let $\epsilon \downarrow 0$ in order to conclude that \hat{f} is in fact a finite Borel measure on \mathbb{R}^d . In particular, this shows that

$$\int_{\mathbb{R}^d} \frac{\hat{f}(dz)}{1 + \|z\|^2} < \infty. \quad (8.11)$$

Thanks to the theory of Dalang [16], condition (8.11) implies the existence of a predictable mild solution to (8.1). Moreover, that solution is unique among all predictable solutions that satisfy

$$\sup_{x \in \mathbb{R}^d} \sup_{t \in [0, T]} \mathbb{E}(|u_t(x)|^k) < \infty \quad \text{for all } T > 0 \text{ and } k \geq 2. \quad (8.12)$$

This solution to (8.1) can also be written in mild form as the a.s.-unique solution to the random integral equation,

$$u_t(x) = (p_t * u_0)(x) + \int_{(0, t) \times \mathbb{R}} p_{t-s}(x - y) u_s(y) \eta(ds dy), \quad (8.13)$$

where the stochastic integral is understood in the sense of Walsh [39], and now p denotes the natural d -dimensional generalization to (6.5):

$$p_t(x) := \frac{e^{-\|x\|^2/2t}}{(2\pi t)^{d/2}} \quad [t > 0, x \in \mathbb{R}^d]. \quad (8.14)$$

Formally speaking, we can let $d := 1$, $\sigma(z) := z$, and $f(z) := \delta_0(z)$ to see that (8.1) is [in this case] one possible extension of (7.1) to higher dimensions. Although δ_0 is not a continuous and bounded function, it is an appropriate limit of such functions. As such, one can derive (7.1)—in this case—using a limiting procedure from the solution to (8.1) with a suitable approximate identity $\{f_\epsilon\}_{\epsilon>0}$ in place of f . See Bertini and Cancrini [5] for the details.

There is a good way to construct examples of positive-definite continuous and bounded functions $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ as follows:

$$f := h * \tilde{h}, \quad (8.15)$$

where $h \in L^2(\mathbb{R}^d)$ is a nonnegative fixed function, and \tilde{h} denotes the reflection of h [see (8.10)]. Thanks to (8.7), we can see that

$$f(0) = \sup_{z \in \mathbb{R}^d} f(z) = \|h\|_{L^2(\mathbb{R}^d)}^2. \quad (8.16)$$

We will restrict attention to such correlation functions f only. In fact, we concentrate on a slightly-smaller class of correlation functions still. Namely, we

will consider only correlation functions f that satisfy (8.15) for a nonnegative $h \in L^2(\mathbb{R}^d)$ that satisfies the following:

$$\limsup_{n \rightarrow \infty} \frac{1}{\log n} \log \left(\int_{\|z\| > n} [h(z)]^2 dz \right) < 0. \quad (8.17)$$

It is possible to write a Feynman–Kac type representation of the solution to (8.1). That representation implies readily that $u_t(x) > 0$ a.s. for all $t > 0$ and $x \in \mathbb{R}^d$. With this remark in place, we have the following, which is the main result of this section.

Theorem 8.1. *Consider the SPDE (8.1) where the spatial correlation function f of the noise satisfies (8.8) and (8.15) for a nonnegative function $h \in L^2(\mathbb{R}^d)$ that satisfies (8.17). Then, with probability one:*

$$\limsup_{\|x\| \rightarrow \infty} \frac{\log u_t(x)}{(\log \|x\|)^{1/2}} = \sqrt{2tf(0)d}, \quad (8.18)$$

and

$$\text{Dim}_H \left\{ x \in \mathbb{R}^d : \|x\| > e^e, u_t(x) \geq e^{\gamma \sqrt{t \log \|x\|}} \right\} = d - \frac{\gamma^2}{2f(0)}. \quad (8.19)$$

Recall that for all sets E , $\text{Dim}_H E < 0$ means that E is bounded. It follows easily from this convention that the lim sup law (8.18) is a consequence of (8.19). Conus et al [12] proved that the lim sup in (8.18) is strictly positive and finite a.s., and the evaluation of the lim sup is contained in the recent work of Chen [8]. We have included (8.18) merely to highlight the fact that the tall peaks of $u_t(x)$ are of order $\exp\{\gamma \sqrt{t \log \|x\|}\}$ for a constant $\gamma > 0$, and hence that (8.19) is indeed a multifractal description of the tall peaks of $u_t(x)$.

The remainder of this section contains the proof of Theorem 8.1.

8.2 Proof of Theorem 8.1: Upper Bound

The proof of Theorem 8.1 proceeds by verifying the conditions of the general theory of §4. Thus, the proof is divided naturally into two parts: Proof of the dimension upper bound; and a separate derivation of the dimension lower bound.

The dimension upper bound will be obtained by verifying Condition (4.5). Our first lemma is essentially a specialization of the proof of Proposition 4.4 of Conus et al [14]. The only new observation is that since here f is continuous, the constants in Proposition 4.4 can be computed explicitly.

Lemma 8.2. *For all $t > 0$,*

$$\lim_{k \rightarrow \infty} \frac{1}{k^2} \log \mathbb{E} (|u_t(x)|^k) = \frac{f(0)t}{2}, \quad (8.20)$$

uniformly for all $x \in \mathbb{R}^d$.

Proof. If $m \geq 1$ is an integer and $k \in [m, m+1)$, then Jensen's inequality assures us that

$$\|u_t(x)\|_{L^m(\Omega)} \leq \|u_t(x)\|_{L^k(\Omega)} \leq \|u_t(x)\|_{L^{m+1}(\Omega)}. \quad (8.21)$$

Therefore, it suffices to prove that the lemma holds where the limit is taken over all integers $k \rightarrow \infty$.

Proposition 4.4 of Conus et al [14] includes the statement that

$$\mathbb{E}(|u_t(x)|^k) \leq e^{k^2 t f(0)/2}, \quad (8.22)$$

for all real $t \geq 0$, $x \in \mathbb{R}^d$, and integers $k \geq 2$.

We develop a corresponding lower bound by following the proof of Proposition 4.4 of [14], but use the additional hypothesis that f is continuous.

According to the Feynman–Kac formula for the moments of the solution to (8.1)—see Conus [11] and Hu and Nualart [23]—the k th moment of the solution to (8.1) has the following representation:

$$\mathbb{E}(|u_t(x)|^k) = \mathbb{E} \left[\exp \left(\frac{1}{2} \sum_{1 \leq i \neq j \leq k} \int_0^t f(X_i(s) - X_j(s)) ds \right) \right], \quad (8.23)$$

where $\{X_i\}_{i=1}^k$ are independent Brownian motions on \mathbb{R}^d .

Choose and fix some $\varepsilon > 0$. There exists $\delta(\varepsilon) > 0$ such that $f(x) \geq f(0) - \varepsilon$ whenever $\|x\| \leq \delta(\varepsilon)/2$. Now let us consider the event

$$\Omega_\varepsilon := \left\{ \omega \in \Omega : \max_{1 \leq j \leq k} \sup_{s \in [0, t]} \|X_j(s)\|(\omega) \leq \delta(\varepsilon) \right\}. \quad (8.24)$$

The moment formula (8.23) and the continuity of f together imply that

$$\begin{aligned} \mathbb{E}(|u_t(x)|^k) &\geq \exp \left\{ \binom{k}{2} (f(0) - \varepsilon) t \right\} \mathbb{P}(\Omega_\varepsilon) \\ &= \exp \left\{ \binom{k}{2} (f(0) - \varepsilon) t \right\} \left[\mathbb{P} \left\{ \sup_{s \in [0, t/|\delta(\varepsilon)|^2]} \|X_1(s)\| \leq 1 \right\} \right]^k, \end{aligned} \quad (8.25)$$

uniformly for all $t \geq 0$ and $x \in \mathbb{R}^d$, and all integers $k \geq 2$. This implies that

$$\liminf_{k \rightarrow \infty: k \in \mathbb{Z}} k^{-2} \log \mathbb{E}(|u_t(x)|^k) \geq (f(0) - \varepsilon) t / 2, \quad (8.26)$$

and readily yields the desired lower bound since $\varepsilon > 0$ were arbitrary. \square

We can now invert moments, exactly as was done in the proof of Theorem 5.2 of Chen [8], in order to deduce the following. We skip the proof since it really follows the proof of [8, Theorem 5.2] almost exactly.

Lemma 8.3. *Fix $t > 0$ and a bounded domain $D \subset \mathbb{R}^d$. Then uniformly for all $x \in \mathbb{R}^d$,*

$$\begin{aligned} \lim_{z \rightarrow \infty} z^{-2} \log \mathbb{P} \{ \log u_t(x) \geq z \} &= \lim_{z \rightarrow \infty} z^{-2} \log \mathbb{P} \left\{ \log \left[\sup_{y \in Q(x,1)} u_t(y) \right] \geq z \right\} \\ &= -(2f(0)t)^{-1}. \end{aligned} \quad (8.27)$$

Lemma 8.3 verifies the conditions of Theorem 4.1, from which we can deduce half of Theorem 8.1 readily.

Proof of Theorem 8.1: Dimension upper bound. Lemma 8.3 verifies Condition (4.5) of Theorem 4.1 with $X(x) := \log u_t(x)$, $z := (2\gamma f(0)t \log s)^{1/2}$ and $b = 2$. It follows from Theorem 4.1 that

$$\text{Dim}_H \left\{ x \in \mathbb{R}^d : \|x\| > e^e, u_t(x) \geq e^{\gamma \sqrt{t \log \|x\|}} \right\} \leq d - \frac{\gamma^2}{2f(0)}, \quad (8.28)$$

almost surely for every $t, \gamma > 0$. This completes the proof of the dimension upper bound in Theorem 8.1. \square

Remark 8.4. Let us record also the fact that the preceding proof required only that $h \geq 0$ is in $L^2(\mathbb{R}^d)$; the extra regularity condition (8.17) was not needed, and (8.28) holds without that extra condition.

8.3 A coupling of the noise

We will have need for a particular construction of η that can be found essentially in the paper by Conus et al [14]. Let ξ denote a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$; that is, ξ is a centered generalized Gaussian random field with covariance measure

$$\text{Cov}[\xi_t(x), \xi_s(y)] = \delta_0(t-s) \cdot \delta_0(x-y) \quad [s, t > 0, x, y \in \mathbb{R}^d]. \quad (8.29)$$

We can construct a cylindrical Brownian motion B , using ξ , as follows: For all $\varphi \in L^2(\mathbb{R}^d)$ and $t > 0$, define

$$B_t(\varphi) := \int_{(0,t) \times \mathbb{R}^d} \varphi(y) \xi(ds dy) \quad \text{and} \quad B_0(\varphi) := 0. \quad (8.30)$$

Then $\{B_t(\varphi)\}_{t \geq 0, \varphi \in L^2(\mathbb{R}^d)}$ is a centered Gaussian random field with

$$\text{Cov}[B_t(\varphi_1), B_t(\varphi_2)] = \min(s, t) \cdot (\varphi_1, \varphi_2)_{L^2(\mathbb{R}^d)}, \quad (8.31)$$

for all $s, t \geq 0$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. Thus, we see that $\{B_t\}_{t \geq 0}$ is a cylindrical Brownian motion on $L^2(\mathbb{R}^d)$ [see Da Prato and Zabczyk [15, §4.3.1]. In particular, we can recover the space-time white noise ξ from B by noticing that

$$\xi_t(x) = \frac{\partial^{d+1}}{\partial t \partial x_1 \cdots \partial x_d} B_t(x), \quad (8.32)$$

where the derivative is understood in the sense of random linear functionals; see Chapter 2 of Gel'fand and Vilenkin [19] for this topic.

We can *subordinate* a large family of Gaussian random fields to the cylindrical Brownian motion $\{B_t\}_{t \geq 0}$ as follows: For all $\varphi, \psi \in L^2(\mathbb{R}^d)$ we define a new centered Gaussian random field $\{B_t^{(\psi)}(\varphi)\}_{t \geq 0, \psi, \varphi \in L^2(\mathbb{R}^d)}$ by setting

$$B_t^{(\psi)}(\varphi) := B_t(\varphi * \tilde{\psi}), \quad (8.33)$$

where we recall $\tilde{\psi}$ denotes the reflection of ψ [see (8.10)].

The random mapping $(\varphi, \psi) \mapsto B^{(\psi)}(\varphi)$ is bilinear, and the covariance structure of each Gaussian process $B^{(\psi)}$ is dictated by

$$\text{Cov} \left[B_t^{(\psi)}(\varphi_1), B_s^{(\psi)}(\varphi_2) \right] = \min(s, t) \cdot \left(\varphi_1, \varphi_2 * \psi * \tilde{\psi} \right)_{L^2(\mathbb{R}^d)}, \quad (8.34)$$

for every $s, t \geq 0$ and $\varphi_1, \varphi_2, \psi \in L^2(\mathbb{R}^d)$. In particular, (8.15) yields

$$\text{Cov} \left[B_t^{(h)}(\varphi_1), B_s^{(h)}(\varphi_2) \right] = \min(s, t) \cdot (\varphi_1, \varphi_2 * f)_{L^2(\mathbb{R}^d)}, \quad (8.35)$$

for every $s, t \geq 0$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. This is another way to say that the weak derivative $\partial_t B_t^{(h)}(x)$ is a particular construction of the noise $\eta_t(x) dt$. Since we are interested only in the law of the solution u to (8.1), and that law is by construction a function of the law of η , we may—and will—change probability space if we have to in order to construct the noise η on the new probability space as follows:

$$\eta_t(x) := \frac{\partial}{\partial t} B_t^{(h)}(x) \quad (t \geq 0, x \in \mathbb{R}^d). \quad (8.36)$$

Thus, we are justified in using the following notation to denote the Wiener integral $\int_{\mathbb{R}_+ \times \mathbb{R}^d} \Phi d\eta$:

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} \Phi_s(y) \eta(ds dy) := \int_{\mathbb{R}_+ \times \mathbb{R}^d} \Phi_s(y) \partial_s B_s^{(h)}(y) dy, \quad (8.37)$$

for all nonrandom functions $\Phi \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$. Thus, in particular, we may—and will—think of the solution to the stochastic PDE (8.1) as the unique solution to the following stochastic integral equation

$$u_t(x) = 1 + \int_{(0, t) \times \mathbb{R}^d} p_{t-s}(y - x) u_s(y) \partial_s B_s^{(h)}(y) dy, \quad (8.38)$$

for all $t > 0$ and $x \in \mathbb{R}^d$, where the stochastic integral is understood in the sense of Walsh [39] and written using the notation introduced earlier.

For us, an advantage of this construction is that, in this way, not only have a construction of $\eta_t(x)$ for our fixed function h , but we have in fact produced a coupling of $\psi \mapsto \partial_t B_t^{(\psi)}(x)$ which is a linear map and agrees with $\eta_t(x)$ when $\psi = h$.

For purposes of comparison, let us mention that the stochastic differential $F^{(h)}(ds dy)$ of Conus et al [14] is the same thing as our mixed random differential $\partial_s B_s^{(h)}(y) dy$.

8.4 Proof of Theorem 8.1: Lower Bound

In order to prove the lower bound on the dimension in Theorem 8.1 we plan to verify Condition (4.34) using an approach that has the same flavor as the proof of the already-developed lower bound of Theorem 7.1. There are some nontrivial differences in the proofs, however. Most notably, since the spatial correlation of the noise of the present section is not in general 0 even when x and y are very far apart, we need to do something more. To combat this issue, we first approximate the function h by a compactly-supported function h_β , and then follow the approach used in the proof of the lower bound of Theorem 7.1. Our proof will follow the ideas of Conus et al [14] loosely.

Define, for all $x \in \mathbb{R}^d$ and $\beta \geq 1$,

$$h_\beta(x) := h(x)\hat{\rho}_\beta(x) \quad \text{where} \quad \hat{\rho}_\beta(x) := \prod_{j=1}^d \left(1 - \frac{|x_j|}{\beta}\right)^+. \quad (8.39)$$

Every function h_β is in $L^2(\mathbb{R}^d)$ and has compact support. In addition, h_β converges to h pointwise as $\beta \rightarrow \infty$. Therefore, we ought to be able to construct approximations of $u_t(x)$ by first approximating the noise $\partial_t B_t^{(h)}(x) dx$ by the noise $\partial_t B_t^{(h_\beta)}(x) dx$ of the previous subsection.

In order to simplify notation, let us write for every $x := (x_1, \dots, x_d) \in \mathbb{R}^d$, $t > 0$, and $\beta \geq 0$,

$$\mathcal{I}_t(x; \beta) := (0, t) \times [x_1 - \beta\sqrt{t}, x_1 + \beta\sqrt{t}] \times \dots \times [x_d - \beta\sqrt{t}, x_d + \beta\sqrt{t}]. \quad (8.40)$$

We can follow the lead of Conus et al [14], and choose and fix $\beta > 0$, and consider the solution $u^{(\beta)}$ to the following stochastic integral equation:

$$u_t^{(\beta)}(x) := 1 + \int_{\mathcal{I}_t(x; \beta)} p_{t-s}(y - x) u_s^{(\beta)}(y) \partial_s B_s^{(h_\beta)}(y) dy. \quad (8.41)$$

The following was pointed out in [14, §5] without proof.

Proposition 8.5. *The stochastic integral equation (8.41) has a predictable solution that is unique among all such solutions that satisfy the following for all real numbers $k \geq 2$:*

$$\sup_{x \in \mathbb{R}^d} \mathbb{E} \left(\left| u_t^{(\beta)}(x) \right|^k \right) \leq 2e^{8k^2 f(0)t} \quad \text{for all } \beta > 0 \text{ and } t \geq 0. \quad (8.42)$$

We will need some of the ingredients of that proof, and the details are not included in [14]; therefore, let us hash out a few of the standard details.

Keep $\beta > 0$ fixed, and define for all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$u_t^{(\beta, 0)}(x) := 1. \quad (8.43)$$

Then we define iteratively the random field $u_t^{(\beta, m)}(x)$ as follows: For all integers $m \geq 0$, reals $t > 0$, and $x \in \mathbb{R}^d$,

$$u_t^{(\beta, m+1)}(x) := 1 + \int_{\mathcal{I}_t(x; \beta)} p_{t-s}(y-x) u_s^{(\beta, m)}(y) \partial_s B_s^{(h_\beta)}(y) dy. \quad (8.44)$$

The proof of Proposition 8.5 requires two *a priori* bounds. The first controls the moments of the Picard iterates.

Lemma 8.6. *Uniformly for all $t \geq 0$, $x \in \mathbb{R}^d$, $k \geq 2$, and integers $m \geq 0$,*

$$\mathbb{E} \left(\left| u_t^{(\beta, m)}(x) \right|^k \right) \leq 2e^{8k^2 f(0)t}. \quad (8.45)$$

Proof. Throughout, we hold $\beta > 0$ fixed and define for all $t > 0$, $m \geq 0$, and $x \in \mathbb{R}^d$,

$$X_t^{(m)}(x) := \int_{\mathcal{I}_t(x; \beta)} p_{t-s}(y-x) u_s^{(\beta, m)}(y) \partial_s B_s^{(h_\beta)}(y) dy, \quad (8.46)$$

if and when the stochastic integral is defined in the sense of Walsh [39]. As an *a priori* part of this proof we will derive moment bounds for $X_t^{(m)}(x)$.

Let

$$f_\beta := h_\beta * \tilde{h}_\beta, \quad (8.47)$$

where h_β was defined in (8.39). The starting point is that a suitable form of the Burkholder–Davis–Gundy inequality [28, Theorem B.1] implies that for all real numbers $k \geq 2$,

$$\begin{aligned} \left\| X_t^{(m)}(x) \right\|_k^2 &\leq 4k \left\| \int_{\mathcal{I}_t(x; \beta)} \mathcal{T}(s, y, z) ds dz dy \right\|_{k/2} \\ &\leq 4k \int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \|\mathcal{T}(s, y, z)\|_{k/2}, \end{aligned} \quad (8.48)$$

where we have used Minkowski's inequality in the last line, and

$$\begin{aligned} \mathcal{T}(s, y, z) &:= p_{t-s}(y-z) p_{t-s}(z-x) \left| u_s^{(\beta, m)}(y) u_s^{(\beta, m)}(z) \right| f_\beta(y-z) \\ &\leq p_{t-s}(y-z) p_{t-s}(z-x) \left| u_s^{(\beta, m)}(y) u_s^{(\beta, m)}(z) \right| f(0), \end{aligned} \quad (8.49)$$

thanks to (8.7) and the elementary fact that $h_\beta \leq h$, whence $f_\beta \leq f$. In particular, by the Cauchy–Schwarz inequality,

$$\|\mathcal{T}(s, y, z)\|_{k/2} \leq e^{2\alpha s} \left[\mathcal{N}_{\alpha, k} \left(u^{(\beta, m)} \right) \right]^2 p_{t-s}(y-z) p_{t-s}(z-x) f(0), \quad (8.50)$$

for every $\alpha > 0$, where for all space-time random fields $\Psi := \{\Psi_s(y)\}_{s \geq 0, y \in \mathbb{R}^d}$,

$$\mathcal{N}_{\alpha, k}(\Psi) := \sup_{y \in \mathbb{R}^d} \sup_{s \geq 0} \left[e^{-\alpha s} \|\Psi_s(y)\|_k \right]. \quad (8.51)$$

We plug the preceding into (8.48) to see that for all $\alpha > 0$,

$$\begin{aligned} \left\| X_t^{(m)}(x) \right\|_k^2 &\leq 4k \left[\mathcal{N}_{\alpha,k} \left(u^{(\beta,m)} \right) \right]^2 f(0) \cdot \int_0^t e^{2\alpha s} ds \\ &\leq \frac{2kf(0)e^{2\alpha t}}{\alpha} \left[\mathcal{N}_{\alpha,k} \left(u^{(\beta,m)} \right) \right]^2 f(0). \end{aligned} \quad (8.52)$$

Multiply both sides by $\exp(-2\alpha t)$ and maximize over $x \in \mathbb{R}^d$ to see that

$$\mathcal{N}_{\alpha,k} \left(X^{(m)} \right) \leq \mathcal{N}_{\alpha,k} \left(u^{(\beta,m)} \right) \sqrt{\frac{2kf(0)}{\alpha}}. \quad (8.53)$$

Since $\Psi \mapsto \mathcal{N}_{\alpha,k}(\Psi)$ is a bona fide norm on random fields, it follows from (8.44) and the triangle inequality that

$$\mathcal{N}_{\alpha,k} \left(u^{(\beta,m+1)} \right) \leq 1 + \mathcal{N}_{\alpha,k} \left(u^{(\beta,m)} \right) \sqrt{\frac{2kf(0)}{\alpha}}. \quad (8.54)$$

The preceding is valid for all $\alpha > 0$. We now select $\alpha := 8kf(0)$ in order to see that, for this special choice,

$$\mathcal{N}_{8kf(0),k} \left(u^{(\beta,m+1)} \right) \leq 1 + \frac{1}{2} \mathcal{N}_{8kf(0),k} \left(u^{(\beta,m)} \right). \quad (8.55)$$

Since $\mathcal{N}_{\alpha,k}(u^{(\beta,0)}) = 1$ for all $\alpha > 0$, induction implies that

$$\mathcal{N}_{8kf(0),k} \left(u^{(\beta,m+1)} \right) \leq 2. \quad (8.56)$$

This is another way to write the lemma. \square

Next we state and prove the second *a priori* bound that is required for the proof of Proposition 8.5.

Lemma 8.7. *Uniformly for all $t \geq 0$, $x \in \mathbb{R}^d$, $k \geq 2$, and integers $m \geq 0$,*

$$\mathbb{E} \left(\left| u_t^{(\beta,m+1)}(x) - u_t^{(\beta,m)}(x) \right|^k \right) \leq \left(\frac{3}{2^m} \right)^k e^{8k^2 f(0)t}. \quad (8.57)$$

Proof. The proof is similar to that of Lemma 8.6. Recall $X^{(m)}$'s from (8.46). By the Burkholder–Davis–Gundy inequality [28, Theorem B.1],

$$\begin{aligned} \left\| X_t^{(m+1)}(x) - X_t^{(m)}(x) \right\|_k^2 &\leq 4k \left\| \int_{\mathcal{I}_t(x;\beta)} \mathcal{Y}(s, y, z) ds dz dy \right\|_{k/2}^2 \\ &\leq 4k \int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \|\mathcal{Y}(s, y, z)\|_{k/2}, \end{aligned} \quad (8.58)$$

where

$$\begin{aligned} \mathcal{Y}(s, y, z) &:= p_{t-s}(y-z)p_{t-s}(z-x)f(0) \\ &\times \left| u_s^{(\beta,m+1)}(y) - u_s^{(\beta,m)}(y) \right| \cdot \left| u_s^{(\beta,m+1)}(z) - u_s^{(\beta,m)}(z) \right|. \end{aligned} \quad (8.59)$$

Recall $\mathcal{N}_{\alpha,k} := \mathcal{N}_{\alpha,k}$ from (8.51). Since

$$\mathcal{N}_{8kf(0),k} \left(u^{(\beta,m+1)} - u^{(\beta,m)} \right) = \mathcal{N}_{8kf(0),k} \left(X^{(m+1)} - X^{(m)} \right), \quad (8.60)$$

we can easily adapt the proof of Lemma 8.6 to see that

$$\mathcal{N}_{8kf(0),k} \left(u^{(\beta,m+1)} - u^{(\beta,m)} \right) \leq \frac{1}{2} \mathcal{N}_{8kf(0),k} \left(u^{(\beta,m)} - u^{(\beta,m-1)} \right). \quad (8.61)$$

Because $u_t^{(\beta,0)}(x) = 1$, Lemma 8.6 implies that

$$\mathcal{N}_{8kf(0),k} \left(u^{(\beta,1)} - u^{(\beta,0)} \right) \leq \mathcal{N}_{8kf(0),k} \left(u^{(\beta,1)} \right) + \mathcal{N}_{8kf(0),k} \left(u^{(\beta,0)} \right) \leq 3. \quad (8.62)$$

Therefore, $\mathcal{N}_{8kf(0),k} \left(u^{(\beta,m+1)} - u^{(\beta,m)} \right) \leq 3 \cdot 2^{-m}$, which is another way to state the lemma. \square

We are now ready to complete the proof of Proposition 8.5.

Proof of Proposition 8.5. Recall $\mathcal{N}_{\alpha,k}$ from (8.51). Lemmas 8.6 and 8.7 together guarantee the existence of a predictable random field $u^{(\beta)}$ such that

$$\mathcal{N}_{8kf(0),k} \left(u^{(\beta)} \right) \leq 2 \quad \text{and} \quad \lim_{m \rightarrow \infty} \mathcal{N}_{8kf(0),k} \left(u^{(\beta,m)} - u^{(\beta)} \right) = 0. \quad (8.63)$$

The proof of Lemma 8.7 also implies that

$$\lim_{m \rightarrow \infty} \mathcal{N}_{8kf(0),k} \left(X^{(m)} - X \right) = 0, \quad (8.64)$$

where $X^{(m)}$ was defined in (8.46), and

$$X_t(x) := \int_{\mathcal{I}_t(x;\beta)} p_{t-s}(y-x) u_s^{(\beta)}(y) \partial_s B_s^{(h_\beta)}(y) dy. \quad (8.65)$$

These remarks together show that $u^{(\beta)}$ solves (8.41). Uniqueness is similar; in fact, the argument that led to (8.61) can be re-iterated in order to imply that if v were any other predictable solution to (8.41) that satisfies $\mathcal{N}_{8kf(0),k}(v) < \infty$, then

$$\mathcal{N}_{8kf(0),k} \left(v - u^{(\beta)} \right) \leq \frac{1}{2} \mathcal{N}_{8kf(0),k} \left(v - u^{(\beta)} \right), \quad (8.66)$$

and hence $\mathcal{N}_{8kf(0),k}(v - u^{(\beta)}) = 0$. \square

Now that we have justified the existence of a good solution to (8.41) we can establish that $u \approx u^{(\beta)}$ when β is large.

Lemma 8.8 (A coupling lemma). *There exists a finite constant $K > 1$ such that for all real numbers $t > 0$ and $\lambda > 1$, and all integers $\beta \geq 1$,*

$$\mathbb{P} \left\{ \left| u_t(x) - u_t^{(\beta, \lfloor \log \beta \rfloor + 1)}(x) \right| > \lambda \right\} \leq K \exp \left(- \frac{(\log \beta + \log \lambda)^2}{Kt} \right), \quad (8.67)$$

uniformly over all $x \in \mathbb{R}^d$.

Proof. According to Lemma 5.3 of Conus et al [14], there exist finite constants $c > 0$ and $b \in (0, 4)$ such that

$$\sup_{x \in \mathbb{R}} \mathbb{E} \left(\left| u_t(x) - u_t^{(\beta, \lfloor \log \beta \rfloor + 1)}(x) \right|^k \right) \leq c e^{c k^2 t - b k \log \beta}, \quad (8.68)$$

valid uniformly for all $x \in \mathbb{R}^d$ and all real numbers $k, \beta \geq 2$ and $t > 0$. This bound and Chebyshev's inequality together yield the following: Uniformly for all reals $t > 0$ and $k, \lambda, \beta \geq 2$, and $x \in \mathbb{R}^d$,

$$\mathbb{P} \left\{ \left| u_t(x) - u_t^{(\beta, \lfloor \log \beta \rfloor + 1)}(x) \right| > \lambda \right\} \leq c e^{c k^2 t - (\log \lambda + b \log \beta) k}. \quad (8.69)$$

The preceding readily implies the lemma, after we optimize over $k \geq 2$. \square

For every $x, y \in \mathbb{R}^d$, let us define

$$D(x, y) := \min_{1 \leq l \leq d} |x_l - y_l|, \quad (8.70)$$

where we recall $|\dots|$ denotes the ℓ^∞ -norm on \mathbb{R}^d . The following lemma is due to Conus, Joseph, Khoshnevisan, and Shiu.

Lemma 8.9 (Conus et al [14, Lemma 5.4]). *Suppose that $t > 0$ is a real number, $\beta \geq 1$ is an integer, and $x^{(1)}, \dots, x^{(m)}$ are points in \mathbb{R}^d such that*

$$D(x^{(i)}, x^{(j)}) > 2\beta (\lfloor \log \beta \rfloor + 1) (1 + \sqrt{t}) \quad \text{when } 1 \leq i \neq j \leq m. \quad (8.71)$$

Then,

$$u_t^{(\beta, \lfloor \log \beta \rfloor + 1)}(x^{(1)}), \dots, u_t^{(\beta, \lfloor \log \beta \rfloor + 1)}(x^{(m)}) \quad (8.72)$$

are independent random variables.

We can now verify the dimension lower bound of Theorem 8.1.

Proof of Theorem 8.1: Dimension lower bound. Choose and fix a time variable $t > 0$. We will appeal to Theorem 4.7, specifically to the general theory of Section 4, using the identifications $X(x) := \log u_t(x)$, $b := 2$, and $S(x) := \exp(x)$.

Let us fix a real number $\delta \in (0, 1)$ and consider an arbitrary collection $\{x^{(i)}\}_{i=1}^{m^d}$ of points that satisfy the following:

- (a) $x^{(i)} \in \mathcal{S}_n \subset \mathbb{R}^d$ for all $1 \leq i \leq m^d$; and
- (b) $D(x^{(i)}, x^{(j)}) \geq \exp(\delta n)$ whenever $1 \leq i \neq j \leq m^d$.

From now on, we set

$$\beta := \exp(n^{2/3}) \quad \text{and} \quad Y_j := Y_{j,n} := \log \left(u_t^{(\beta, \lfloor \log \beta \rfloor + 1)}(x^{(j)}) \right), \quad (8.73)$$

for all $1 \leq i \leq m^d$. We might observe that there exists $N := N_t > 0$ such that

$$2(\lfloor \log \beta \rfloor + 1)\beta(1 + \sqrt{t}) < e^{\delta n} \quad \text{for all } n \geq N. \quad (8.74)$$

Therefore, according to Lemma 8.9, Y_1, \dots, Y_{m^d} are independent whenever $n \geq N$; and Lemma 8.8 ensures that

$$\max_{1 \leq j \leq m} \mathbb{P} \left\{ \left| S(X(x^{(j)})) - S(Y_j) \right| > 1 \right\} \leq K \exp \left(-\frac{n^{4/3}}{Kt} \right). \quad (8.75)$$

Since the constant K does not depend on the choice of $x^{(1)}, \dots, x^{(m)} \in \mathcal{S}_n$, we have shown that the coupling Condition (4.34) holds, with room to spare. Therefore, Theorem 4.7 implies that a.s.,

$$\text{Dim}_{\mathbb{H}} \left\{ x \in \mathbb{R}^d : \|x\| > e^e, u_t(x) \geq e^{\gamma \sqrt{t \log \|x\|}} \right\} \geq d - \frac{\gamma^2}{2f(0)}, \quad (8.76)$$

for all $\gamma > 0$. In light of the already-proved upper bound this completes our proof of Theorem 8.1. \square

Acknowledgement. We thank Professor Gregory Lawler heartily for his many insightful remarks, questions, and comments that ultimately led us to this enjoyable research problem.

References

- [1] Alberverio, Sergio, Stanislav Molchanov, and Donatas Surgailis. Stratified structure of the universe and Burgers' equation — a probabilistic approach, *Probab. Th. Rel. Fields* **100** (1994) 457–484.
- [2] Albin, J. M. P. and H. Choi. A new proof of an old result by Pickands, *Electr. Comm. in Probab.* **15** (2010) 339–345.
- [3] Barlow, M. T., and S. J. Taylor. Fractional dimension of sets in discrete spaces, *J. Phys. A* (3) **64** (1989) 2621–2626.
- [4] Barlow, Martin T., and S. James Taylor. Defining fractal subsets of \mathbb{Z}^d , *Proc. London Math. Soc.* (3) **64** (1992) 125–152.
- [5] Bertini, Lorenzo and Nicoletta Cancrini. The stochastic heat equation: Feynman–Kac formula and intermittence, *J. Statist. Physics* **78**(5/6) (1994) 1377–1402.
- [6] Borodin, Alexei and Corwin, Ivan, Moments and Lyapunov exponents for the parabolic Anderson model, *Ann. Appl. Probab.* **24**(33) (2014) 1172–1198.
- [7] Carmona, René A. and S. A. Molchanov. Parabolic Anderson Problem and Intermittency, *Memoires of the Amer. Math. Soc.* **108**, American Mathematical Society, Rhode Island, 1994.
- [8] Chen, Xia. Spatial asymptotics for the parabolic Anderson models with generalized time-space Gaussian noise, submitted, 2014.
- [9] Chen, Xia. *Random Walk Intersections: Large Deviations and Related Topics*, American Mathematical Society, Providence, RI, 2010.

- [10] Collela, Phillip, and Oscar E. Lanford. *Appendix: Sample Field behavior for the free Markov random field*, In: “Constructive Quantum Field Theory” (G. Velo and A. S. Wightman, ed’s) Lecture Notes in Physics, Vol. 25, pp. 44–70, 1973.
- [11] Conus, Daniel. Moments for the parabolic Anderson model: On a result of Hu and Nualart, *Comm. Stoch. Analysis* **7**(1) (2013) 125–152.
- [12] Conus, Daniel, Mathew Joseph and Davar Khoshnevisan. On the chaotic character of the stochastic heat equation, before the onset of intermittency, *Ann. Probab.* **41**(3B) (2013) 2225–2260.
- [13] Conus, Daniel, Mathew Joseph and Davar Khoshnevisan. Correlation-length bounds, and estimates for intermittent islands in parabolic SPDEs, *Electr. J. Probab.* **17**(102) (2013) (15 pp).
- [14] Conus, Daniel, Mathew Joseph, Davar Khoshnevisan, and Shang-Yuan Shiu. On the chaotic character of the stochastic heat equation, II, *Probab. Th. Rel. Fields* **156** (2013) 483–533.
- [15] Da Prato, Giuseppe and Jerzy Zabczyk. *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, UK, 1992.
- [16] Dalang, Robert C. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.’s, *Electron. J. Probab.* **4**(6) (1999) 29 pp. (electronic).
- [17] Dalang, Robert, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. *A Minicourse in Stochastic Partial Differential Equations* (2006). In: Lecture Notes in Mathematics, vol. 1962 (D. Khoshnevisan and F. Rassoul-Agha, editors) Springer-Verlag, Berlin, 2009.
- [18] Foondun, Mohammud, and Davar Khoshnevisan. Intermittence and nonlinear stochastic partial differential equations, *Electr. J. Probab.* **14**(2) (2009) 548–568.
- [19] Gel’fand, I. M. and N. Ya. Vilenkin. *Generalized Functions*, Vol. 4, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977], Applications of harmonic analysis, Translated from the Russian by Amiel Feinstein.
- [20] Gibbon, J. D. and E. S. Titi. Cluster formation in complex multi-scale systems, *Proc. R. Soc. A* **461** (2005) 3089–3097.
- [21] Hairer, Martin. Solving the KPZ equation, *Ann. Math.* **178**(2) (2013) 559–664.
- [22] Harper, Adam J. Pickand’s constant H_α does not equal $1/\Gamma(1/\alpha)$, for small α . Available at <http://arxiv.org/pdf/1404.5505v1.pdf>
- [23] Hu, Yaozhong, and David Nualart. Stochastic heat equation driven by fractional noise and local time, *Probab. Th. Rel. Fields* **143**(1–2) (2009) 285–328.
- [24] Joseph, Mathew, Davar Khoshnevisan, and Carl Mueller. Strong invariance and noise comparison principles for some parabolic SPDE, submitted, 2014, preprint available at <http://arxiv.org/abs/1404.6911>.
- [25] Kardar, Mehran. Replica Beth ansatz studies of two-dimensional interfaces with quenched random impurities, *Nucl. Physics* **B290** [FS20] (1987) 582–602.
- [26] Kardar, Mehran, Parisi, Giorgio, and Zhang, Yi-Cheng. Dynamic scaling of growing interfaces, *Phys. Rev. Lett.* **56** (1986) 889–892.
- [27] Kardar, Mehran and Yi-Cheng Zhang. Scaling of directed polymers in random media, *Phys. Rev. Lett.* **58**(20) (1987) 2087–2090.

- [28] Khoshnevisan, Davar. *Analysis of Stochastic Partial Differential Equations*, CBMS Regional Conference Series in Mathematics, 119. American Mathematical Society, Providence, RI, 2014.
- [29] Motoo, Minoru. Proof of the law of the iterated logarithm through diffusion equation, *Ann. Instit. Statist. Math.* **10**(1) (1959) 21–28.
- [30] Mueller, Carl. On the support of solutions to the heat equation with noise, *Stochastics and Stochastics Rep.* **37**(4) (1991) 225–245.
- [31] Mueller, Carl and David Nualart. Regularity of the density for the stochastic heat equation, *Electr. J. Probab.* **13**(74) (2008) 2248–2258.
- [32] Naudts, J. Dimension of discrete fractal spaces, *J. Phys. A* **21** (1988) 447–452.
- [33] Orey, Steven, and William E. Pruitt. Sample functions of the N-parameter Wiener process, *Ann. Probab.* **1**(1) (1973) 138–163.
- [34] Paladin, G., L. Peliti, and A. Vulpiani. Intermittency as multifractality in history space, *J. Phys. A* **19** (1986) no. 16, L991–996.
- [35] Paley, R. E. A. C., and A. Zygmund. A note on analytic functions on the circle, *Proc. Cambridge Phil. Soc.* **28**(3) (1932) 266–272.
- [36] Pickands, James, III. Upcrossing probabilities for stationary Gaussian processes, *Trans. Amer. Math. Soc.* **145** (1969) 51–73.
- [37] Qualls, Clifford, and Hisao Watanabe. As asymptotic 0-1 behavior of Gaussian process, *Ann. Math. Statist.* **42**(6) (1971) 2029–2035.
- [38] Strassen, V. An invariance principle for the law of the iterated logarithm, *Zeit. für Wahr. verw. Geb.* **3** (1964) 211–226.
- [39] Walsh, John B. *An Introduction to Stochastic Partial Differential Equations*, in: École d’été de probabilités de Saint-Flour, XIV—1984, 265–439, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986.
- [40] Weber, M. Some examples of application of the metric entropy method, *Acta Math. Hungar.* **105**(1-2) (2004) 39–83.

Davar Khoshnevisan & Kunwoo Kim

(davar@math.utah.edu & kkim@math.utah.edu)

Dept. Mathematics, University of Utah, Salt Lake City, UT 84112-0090

Yimin Xiao

(xiao@stt.msu.edu)

Dept. Statistics & Probability, Michigan State University, East Lansing, MI 48824